GEGENBAUER POLYNOMIALS REVISITED

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1. INTRODUCTION

The *Gegenbauer* (or *ultraspherical*) *polynomials* $C_n^\lambda(x)$ $(\lambda > -\frac{1}{2},\ |x| \leqslant 1)$ are defined by

$$C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x \tag{1.1}$$

with the recurrence relation

$$nC_n^\lambda(x) = 2x(\lambda + n - 1)C_{n-1}^\lambda(x) - (2\lambda + n - 2)C_{n-2}^\lambda(x) \quad (n \geqslant 2). \tag{1.2}$$

Gegenbauer polynomials are related to $T_n(x)$, the *Chebyshev polynomials of the first kind*, and to $U_n(x)$, the *Chebyshev polynomials of the second kind*, by the relations

$$T_n(x) = \frac{n}{2}\lim_{\lambda \to 0}\left(\frac{C_n^\lambda(x)}{\lambda}\right) \quad (n \geqslant 1), \tag{1.3}$$

and

$$U_n(x) = C_n^1(x). \tag{1.4}$$

Properties of the rising and descending diagonals of the Pascal-type arrays of $\{T_n(x)\}$ and $\{U_n(x)\}$ were investigated in [2], [3], and [5], while in [4] the rising diagonals of the similar array for $C_n^\lambda(x)$ were examined.

Here, we consider the descending diagonals in the Pascal-type array for $\{C_n^\lambda(x)\}$, with a backward glance at some of the material in [4].

As it turns out, the descending diagonal polynomials have less complicated computational aspects than the polynomials generated by the rising diagonals.

Brief mention will also be made of the generalized Humbert polynomial, of which the Gegenbauer polynomials and, consequently, the Chebyshev polynomials, are special cases.

2. DESCENDING DIAGONALS FOR THE GEGENBAUER POLYNOMIAL ARRAY

Table 1 sets out the first few Gegenbauer polynomials (with $y = 2x$):

TABLE 1. Descending Diagonals for Gegenbauer Polynomials

$$\left\{\begin{array}{lll}
C_0^\lambda(x) &= 1 & \qquad d_0^\lambda(x)\\
C_1^\lambda(x) &= \lambda y & \\
C_2^\lambda(x) &= \dfrac{(\lambda)_2 y^2}{2!} - \lambda & \qquad d_1^\lambda(x)\\
C_3^\lambda(x) &= \dfrac{(\lambda)_3 y^3}{3!} - \dfrac{(\lambda)_2 y}{1!} & \\
C_4^\lambda(x) &= \dfrac{(\lambda)_4 y^4}{4!} - \dfrac{(\lambda)_3 y^2}{2!} + \dfrac{(\lambda)_2}{2!} & \qquad d_2^\lambda(x)\\
C_5^\lambda(x) &= \dfrac{(\lambda)_5 y^5}{5!} - \dfrac{(\lambda)_4 y^3}{3!} + \dfrac{(\lambda)_3 y}{2!} & \\
C_6^\lambda(x) &= \dfrac{(\lambda)_6 y^6}{6!} - \dfrac{(\lambda)_5 y^4}{4!} + \dfrac{(\lambda)_4 y^2}{2!2!} - \dfrac{(\lambda)_3}{3!} & \qquad d_3^\lambda(x)\\
C_7^\lambda(x) &= \dfrac{(\lambda)_7 y^7}{7!} - \dfrac{(\lambda)_6 y^5}{5!} + \dfrac{(\lambda)_5 y^3}{2!3!} - \dfrac{(\lambda)_4 y}{3!} & \\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &
\end{array}\right. \tag{2.1}$$

wherein we have written

$$(\lambda)_n = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1). \tag{2.2}$$

Polynomials (2.1) may be obtained either from the generating recurrence (1.2) together with the initial values (1.1), or directly from the known explicit summation representation

$$C_n^\lambda(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m} (2x)^{n-2m}}{m!(n - 2m)!}, \quad \lambda \text{ an integer and } n \geqslant 2, \tag{2.3}$$

where, as usual, $[n/2]$ symbolizes the integer part of $n/2$.

The generating function for the Gegenbauer polynomials is

$$\sum_{n=0}^{\infty} C_n^\lambda(x) t^n = (1 - 2xt + t^2)^{-\lambda} \qquad (|t| < 1). \tag{2.4}$$

Designate the descending diagonals in Table 1, indicated by lines, by the symbols $d_j^\lambda(x)$ $(j = 0, 1, 2, \ldots)$.

Then we have

$$\begin{aligned} &d_0^\lambda(x) = 1, \quad d_1^\lambda(x) = \lambda(2x - 1), \quad d_2^\lambda(x) = \frac{(\lambda)_2 (2x - 1)^2}{2!} \\ &d_3^\lambda(x) = \frac{(\lambda)_3 (2x - 1)^3}{3!}, \quad d_4^\lambda(x) = \frac{(\lambda)_4 (2x - 1)^4}{4!}, \ \ldots . \end{aligned} \tag{2.5}$$

From the emerging pattern in (2.5), one can confidently expect that

$$d_n^\lambda(x) = \frac{(\lambda)_n(2x - 1)^n}{n!} = \binom{\lambda + n - 1}{n}(2x - 1)^n, \tag{2.6}$$

a result which we now proceed to prove.

Proof of (2.6): Suppose we represent the pairs of values of m and n which give rise to $d_n^\lambda(x)$ by the couplet (m, n).

Then, at successive "levels" of the descending diagonal $d_n^\lambda(x)$ in Table 1, we have the couplets

$$(0, n), (1, n + 1), (2, n + 2), \ldots, (n - 1, 2n - 1), (n, 2n),$$

so that corresponding values of $n - 2m$ are $n, n - 1, n - 2, \ldots, 1, 0$, while $n - m$ always has the value n. [It is important to note that the maximum value for m in the couplets must be n.]

Consequently, from Table 1 and (2.3), with $y = 2x$ for convenience, we have

$$\begin{aligned} d_n^\lambda(x) &= \frac{(\lambda)_n y^n}{0!n!} - \frac{(\lambda)_n y^{n-1}}{1!(n-1)!} + \frac{(\lambda)_n y^{n-2}}{2!(n-2)!} - \cdots + (-1)^n \frac{(\lambda)_n y^0}{n!0!} \\ &= \frac{(\lambda)_n}{n!}\left\{\binom{n}{0}y^n - \binom{n}{1}y^{n-1} + \binom{n}{2}y^{n-2} - \cdots + (-1)^n\binom{n}{n}y^0\right\} \\ &= \frac{(\lambda)_n}{n!}(y - 1)^n = \frac{(\lambda)_n}{n!}(2x - 1)^n = \binom{\lambda + n - 1}{n}(2x - 1)^n. \end{aligned}$$

From (2.6) it follows immediately that

$$\frac{d_n^\lambda(x)}{d_{n-1}^\lambda(x)} = \frac{\lambda + n - 1}{n}(2x - 1), \quad n \geqslant 1. \tag{2.7}$$

Moreover,

$$(2x - 1)\frac{d}{dx}(d_n^\lambda(x)) - 2nd_n^\lambda(x) = 0, \quad n \geqslant 0, \tag{2.8}$$

readily follows.

Putting

$$g \equiv d(\lambda, x, t) = \sum_{n=0}^{\infty} d_n^\lambda(x)t^n, \tag{2.9}$$

we find that the generating function for $\{d_n^\lambda(x)\}$ is

$$g = [1 - (2x - 1)^t]^{-\lambda}. \tag{2.10}$$

Furthermore,

$$2t\frac{\partial g}{\partial t} - (2x - 1)\log(2x - 1)\frac{\partial g}{\partial x} = 0, \tag{2.11}$$

which is independent of λ.

Additionally, we easily establish that

$$2\lambda^2 t(2x - 1)^{t-1}\frac{\partial g}{\partial x} - g^{-\lambda^{-1}}\log g\,\frac{\partial g}{\partial x} = 0 \tag{2.12}$$

and

$$\lambda^2(2x - 1)^t \log(2x - 1)\frac{\partial g}{\partial \lambda} - g^{-\lambda^{-1}} \log g \frac{\partial g}{\partial t} = 0 \tag{2.13}$$

if we allow λ to vary.

Differentiating (2.8) w.r.t. x and substituting from (2.8), we obtain

$$(2x - 1)^2\frac{d^2}{dx^2}(d_n^\lambda(x) - 2^2n(n - 1)d_n^\lambda(x) = 0. \tag{2.14}$$

Continued repetition of this process, with substitution from the previous steps, ultimately yields

$$(2x - 1)^r\frac{d^r}{dx^r}(d_n^\lambda(x)) - 2^r r!\binom{n}{r}d_n^\lambda(x) = 0 \tag{2.15}$$

for the r^{th} derivative of the descending diagonal polynomial. If we write $z = d_0^\lambda(x)$ for simplified symbolism, result (2.15) appears in a more attractive visual form as, when $r = n$,

$$(2x - 1)^n z^{(n)} - 2^n n! z = 0 \tag{2.15$'$}$$

or by (2.6),

$$z^{(n)} - 2^n(\lambda)_n = 0. \tag{2.15$''$}$$

Observe that (2.15) can also be expressed as

$$\frac{d^r}{dx^r}(d_n^\lambda(x)) = 2^r(\lambda)_r d_{n-r}^{\lambda+r}(x), \quad r = 1, 2, \ldots, n, \tag{2.15$'''$}$$

on using $(\lambda)_n(\lambda + r)_{n-r} = (\lambda)_n$ and (2.6).

Note the formal equivalence of (2.15)$'''$ and the known differential equation for Gegenbauer polynomials

$$\frac{d^r}{dx^r}C_n^\lambda(x) = 2^r(\lambda)_r C_{n-r}^{\lambda+r}(x). \tag{2.16}$$

Elementary calculations yield, additionally, by using (2.8) and (2.7),

$$(2x - 1)\frac{d}{dx}(d_n^\lambda(x)) = \frac{n}{\lambda + n}\frac{d}{dx}(d_{n+1}^\lambda(x)), \tag{2.17}$$

which differs in form from the corresponding result involving Gegenbauer polynomials.

3. SPECIAL CASES: CHEBYSHEV POLYNOMIALS

If we substitute $\lambda = 1$ in the relevant results of the preceding section we obtain corresponding results already given in [3] for the special case (1.4) of the Chebyshev polynomials $U_n(x)$. [Allowance must be made for a small variation in notation, namely $d_n^1(x) = b_{n+1}(x)$ in [3]; e.g., $d_4^1(x) = (2x - 1)^4 = b_5(x)$.]

Coming now to the similar results for the Chebyshev polynomials $T_n(x)$, we appreciate that the limiting process (1.3) requires a less obvious approach.

Let us write

$$\mathscr{D}_n^\lambda(x) = (n + 1)d_{n+1}^\lambda(x) - nd_n^\lambda(x) \qquad (n \geqslant 0). \tag{3.1}$$

By careful analogy with the forms of (1.3), we may then define

$$D_n(x) = \frac{1}{2}\lim_{\lambda\to 0}\left(\frac{\mathscr{D}_n^\lambda(x)}{\lambda}\right), \tag{3.2}$$

where $D_n(x)$ is the n^{th} descending diagonal polynomial for $T_n(x)$.

Calculation yields

$$D_n(x) = \frac{1}{2}(2x - 2)(2x - 1)^n = (x - 1)(2x - 1)^n. \tag{3.3}$$

Comparison should not be made with corresponding results produced in [3] where, it should be noted, each Chebyshev function is *twice* the corresponding Chebyshev polynomial in this paper. Accordingly, we have $2O_m(x) = a_{n+2}(x)$ in [3]; e.g., $D_4(x) = (x - 1)(2x - 1)^4 = (1/2)a_6(x)$.

Thus, we have shown that our results for the descending diagonals in the Pascal-type array of Gegenbauer polynomials are generalizations of corresponding results for the specialized Chebyshev polynomials, as expected.

4. GENERALIZED HUMBERT POLYNOMIALS

Along with many other polynomials, the Gegenbauer polynomials (and consequently the Chebyshev polynomials) are special cases of the *generalized Humbert polynomial* (see Gould [1]).

Generalized Humbert polynomials, which are represented by the symbol

$$P_n(m, x, y, p, C)$$

are defined by the generating function

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n, \tag{4.1}$$

where $m \geqslant 1$ is an integer and the other parameters are in general unrestricted.

Particular cases of the generalized Humbert polynomial are:

$$\begin{cases} P_n(2, x, 1, -\frac{1}{2}, 1) = P_n(x) & (\textit{Legendre}, 1784) \\ P_n(2, x, 1, -1, 1) = U_n(x) & (\textit{Chebyshev}, 1859) \\ P_n(2, x, 1, -\lambda, 1) = C_n^{\lambda}(x) & (\textit{Gegenbauer}, 1874) \\ P_n(3, x, 1, -\frac{1}{2}, 1) = \mathscr{P}_n(x) & (\textit{Pincherle}, 1890) \\ P_n(m, x, 1, -\nu, 1) = \Pi_{n,m}^{\nu}(x) & (\textit{Humbert}, 1921) \\ P_n(2, x, -1, -1, 1) = \phi_{n+1}(x) & (\textit{Byrd}, 1963) \\ P_n\left(m, x, 1, -\frac{1}{m}, 1\right) = P_n(m, x) & (\textit{Kinney}, 1963) \end{cases} \tag{4.2}$$

The recurrence relation for the generalized Humbert polynomial is

$$CnP_n - m(n - 1 - p)xP_{n-1} + (n - m - mp)yP_{n-m} = 0 \quad (n \geqslant m \geqslant 1), \tag{4.3}$$

where we have written $P_n = P_n(m, x, y, p, C)$ for brevity.

Suitable substitution of the parameters in (4.2) for Gegenbauer polynomials reduces (4.3) to (1.2).

In passing, it might be noted in (4.2) that Legendre polynomials are special cases of Gegenbauer polynomials occurring when $\lambda = \frac{1}{2}$. Hence, results for Gegenbauer polynomials $C_n^{\lambda}(x)$ in [4] and in this article may be specialized for the Legendre polynomials $C_n^{\frac{1}{2}}(x)$. Moreover, Gegenbauer polynomials are closely related to Jacobi polynomials, and they may also be expressed in terms of hypergeometric functions.

Using the generating function for Fibonacci numbers F_n, namely

$$(1 - x - x^2)^{-1} = \sum_{n=1}^{\infty} F_n x^{n-1}, \tag{4.4}$$

we readily see that

$$P_n(2, \tfrac{1}{2}, -1, -1, 1) = F_{n+1} = \sum_{k=0}^{[n/2]} \binom{n-k}{k}, \tag{4.5}$$

whence the recurrence relation (4.3) simplifies to the defining recurrence relation

$$F_{n+1} - F_n - F_{n-1} = 0. \tag{4.6}$$

Gould [1] remarks that Eq. (4.5) is better than the usual device of using Chebyshev or other polynomials with imaginary exponent for expressing Fibonacci numbers.

While it is not the purpose of this paper to pursue the properties of the generalized Humbert polynomial, it is thought that publicizing their connection with the polynomials under discussion—Gegenbauer and Chebyshev—may be a useful service.

To whet the appetite of the interested reader for further knowledge of the generalized Humbert polynomial, we append the explicit form given in [1]:

$$P_n(m, x, y, p, C) = \sum_{k=0}^{[n/m]} \binom{p}{k}\binom{p-k}{n-m-k} C^{p-n+(m-1)k} y^k (-mx)^{n-mk}, \tag{4.7}$$

from which one may obtain the explicit forms of the special cases given in (4.2).

Likewise, the first few terms of the polynomials in (4.2) may be checked against the generalized terms given in [1]:

$$\begin{cases} P_0 = C^p \\ P_1 = -pmxC^{p-1} + p\binom{p-1}{1-m} y(-mx)^{1-m} C^{p+m-2} \\ P_2 = \binom{p}{2} C^{p-2} m^2 x^2 + p\binom{p-1}{2-m} C^{p+m-3} y(-mx)^{2-m} \end{cases} \tag{4.8}$$

with

$$P_n = \binom{p}{n} C^{p-n}(-mx)^n \qquad (m > n). \tag{4.9}$$

REFERENCES

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