ON BERNSTEIN'S COMBINATORIAL IDENTITIES

MARIE-LOUIS LAVERTU & CLAUDE LEVESQUE
Université Laval, Québec, P. Québec, Canada G1K 7P4

(Submitted February 1984)

*To the memory of Professor Dr. Leon Bernstein**

0. INTRODUCTION

Using elementary properties of algebraic numbers of certain finite extensions of $\mathbb{Q}$, L. Bernstein obtained in [1], [2], [3], [4], and [5] some combinatorial identities. In this paper, we want to give a clear and quick matrix treatment of Bernstein's technique, from which it will be seen that his combinatorial identities are in fact determinants.

In Section 1, writing the powers of an algebraic number ω of degree m over $\mathbb{Q}$ as

$$\omega^n = r_{1n} + r_{2n}\omega + \cdots + r_{mn}\omega^{m-1},$$

we give, in (1.4) and (1.5), the m^{th}-order linear recurrences satisfied by the numbers

$$r_{jn}, \ n \in \mathbb{Z}, \ j = 1, 2, \ldots, m.$$

Let us note that L. Bernstein is always considering the case where $j = 1$ and ω is a unit of $\mathbb{Q}(\omega)$: see [3]; as far as [1], [2], [4], and [5] are concerned, L. Bernstein deals with the case $m = 3$.

In Section 2, Euler's generating functions are calculated in two ways: one with the help of the sums p_t of all symmetric functions of weight t; the other using the multinomial theorem. The second method is used by L. Bernstein, but the concluding remark of the last paragraph still applies.

A very general procedure combining the properties of the norm of an algebraic integer and Cramer's rule is described in Section 3, which leads to what can be called *combinatorial identities*.

In Section 4, we conclude this paper by giving a formula for r_{jn} involving the determinant of a Vandermonde matrix and the determinant of a matrix that is "almost" of the Vandermonde type.

1. RECURRENCE RELATIONS

Let ω be a root of the polynomial

$$f(X) = X^m + k_1X^{m-1} + \cdots + k_{m-1}X + k_m = (X - \alpha_1)(X - \alpha_2) \ldots (X - \alpha_m)$$

irreducible over $\mathbb{Q}$ with m distinct (nonzero) roots $\alpha_1 = \omega, \alpha_2, \ldots, \alpha_m$, whence the field $\mathbb{Q}(\omega)$ is of degree m over $\mathbb{Q}$. Let us consider the (positive, negative, zero) powers of ω.

Professor L. Bernstein died on March 12, 1984, of a cerebral hemorrhage.

For $n \in \mathbb{Z}$, let

$$\omega^n = r_{1n} + r_{2n}\omega + \cdots + r_{mn}\omega^{m-1}$$

with coefficients in $\mathbb{Q}$. Since

$$\omega^m = -k_1\omega^{m-1} - \cdots - k_{m-1}\omega - k_m,$$

we obtain the equality

$$\omega^{n+1} = -k_m r_{mn} + (r_{1n} - k_{m-1}r_{mn})\omega + (r_{2n} - k_{m-2}r_{mn})\omega^2 + \cdots + (r_{m-1,\,n} - k_1 r_{mn})\omega^{m-1},$$

which leads to the system

$$\begin{cases} r_{1,\,n+1} = 0r_{1n} + 0r_{2n} + \cdots + 0r_{m-1,\,n} - k_m r_{mn}, \\ r_{2,\,n+1} = 1r_{1n} + 0r_{2n} + \cdots + 0r_{m-1,\,n} - k_{m-1} r_{mn}, \\ r_{3,\,n+1} = 0r_{1n} + 1r_{2n} + \cdots + 0r_{m-1,\,n} - k_{m-2} r_{mn}, \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \\ r_{m,\,n+1} = 0r_{1n} + 0r_{2n} + \cdots + 1r_{m-1,\,n} - k_1 r_{mn}. \end{cases} \tag{1.1}$$

Define the matrices

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & -k_m \\ 1 & 0 & \dots & 0 & -k_{m-1} \\ 0 & 1 & \dots & 0 & -k_{m-2} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -k_1 \end{bmatrix}, \qquad R_n = \begin{bmatrix} r_{1n} \\ r_{2n} \\ r_{3n} \\ \vdots \\ r_{mn} \end{bmatrix},$$

$$\Omega = [1 \quad \omega \quad \omega^2 \quad \dots \quad \omega^{m-1}], \qquad I_m = \text{Identity matrix},$$

of dimension $m \times m$, $m \times 1$, $1 \times m$, $m \times m$, respectively.

Hence, we have $\omega^n = \Omega R_n$ and

$$R_n = I_m R_n, \quad R_{n+1} = CR_n, \quad \dots, \quad R_{n+t} = C^t R_n, \quad \dots,$$

from which we conclude

$$R_n = C^n R_0 \quad \text{with } R_0 = [1 \quad 0 \quad \dots \quad 0]^t,$$

and

$$r_{jn} = (j,\ 1) \text{ element of } C^n. \tag{1.2}$$

It is worth noting that system (1.1) leads to the following matrix, which means that it suffices to have a formula for r_{1t} in order to know all the coefficients of ω^n:

$$R_n = \begin{bmatrix} r_{1n} \\ r_{1,\,n-1} + \frac{k_{m-1}}{k_m} r_{1n} \\ r_{1,\,n-2} + \frac{k_{m-1}}{k_m} r_{1,\,n-1} + \frac{k_{m-2}}{k_m} r_{1n} \\ \vdots \\ r_{1,\,n+1-m} + \frac{k_{m-1}}{k_m} r_{1,\,n+2-m} + \cdots + \frac{k_1}{k_m} r_{1n} \end{bmatrix}.$$

It is well known that the characteristic equation of C is

$$\lambda^m + k_1\lambda^{m-1} + \cdots + k_{m-1}\lambda + k_m,$$

whereupon the eigenvalues of C are $\alpha_1, \alpha_2, \ldots, \alpha_m$. Since

$$C^m = -k_1C^{m-1} - \cdots - k_{m-1}C - k_mI_m,$$

we deduce

$$R_{n+m} = C^mR_n = -k_1R_{n+m-1} - \cdots - k_{m-1}R_{n+1} - k_mR_n,$$

from which we conclude (with $1 \leqslant j \leqslant m$):

$$r_{jn} = -k_1r_{j,\,n-1} - \cdots - k_{m-1}r_{j,\,n-m+1} - k_mr_{j,\,n-m}. \tag{1.3}$$

In particular we obtain, for the coefficients of ω^t and ω^{-t} with $t \geqslant m$, the two following m^{th}-order linear recurrences (with $1 \leqslant j \leqslant m$):

$$r_{jt} = -k_1r_{j,\,t-1} - \cdots - k_{m-1}r_{j,\,t-m+1} - k_mr_{j,\,t-m}, \tag{1.4}$$

$$r_{j,\,-t} = -\frac{k_{m-1}}{k_m} r_{j,\,-t+1} - \cdots - \frac{k_1}{k_m} r_{j,\,-t+m-1} - \frac{1}{k_m} r_{j,\,-t+m}, \tag{1.5}$$

with the initial conditions for $0 \leqslant i \leqslant m-1$ being

$$r_{ji} = (j,\ 1) \text{ element of } C^i = \begin{cases} 1 & \text{if } j = i+1, \\ 0 & \text{elsewhere,} \end{cases} \tag{1.6}$$

$$r_{j,\,-i} = (j,\ 1) \text{ element of } C^{-i}. \tag{1.7}$$

Note that for the rest of this article, as opposed to [1], [2], [3], [4], and [5], we do not restrict ourselves to the case $j = 1$.

2. GENERATING FUNCTIONS

Using the m^{th}-order linear recurrence given in (1.4) and the known values of r_{ji} in (1.6) we obtain, for $j = 1, \ldots, m$,

$$\left(\sum_{n=0}^{\infty} r_{jn}X^n\right)(1 + k_1X + \cdots + k_{m-1}X^{m-1} + k_mX^m) = X^{j-1} + k_1X^j + \cdots + k_{m-j}X^{m-1}.$$

So

$$\sum_{n=0}^{\infty} r_{jn} X^n = \frac{X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}}{1 + k_1 X + \cdots + k_{m-1} X^{m-1} + k_m X^m}$$
$$= \frac{X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}}{(1 - \alpha_1 X)(1 - \alpha_2 X) \ldots (1 - \alpha_m X)}$$
$$= (X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1})(1 + p_1 X + p_2 X^2 + \cdots + p_t X^t + \cdots),$$

where p_t stands for the sum of all symmetric functions of weight t in $\alpha_1, \ldots, \alpha_m$. Hence, we conclude

$$r_{jn} = \begin{cases} \sum_{t=0}^{m-j} k_t p_{n-j+1-t} & \text{if } n \geqslant j - 1, \\ 0 & \text{if } 0 \leqslant n < j - 1, \end{cases} \tag{2.1}$$

where, as stated in [6],

$$p_t = \sum_{i=1}^{m} \frac{\alpha_i^{m-1+t}}{f'(\alpha_i)}, \text{ with } p_0 = k_0 = 1,\ p_{-1} = p_{-2} = \cdots = p_{-m+1} = 0.$$

Similarly, using (1.5), (1.3), and then (1.6), we obtain

$$\left(\sum_{n=0}^{\infty} r_{j,-n} X^n\right)(k_m + k_{m-1}X + \cdots + k_1 X^{m-1} + X^m)$$
$$\begin{aligned} = {} & k_m r_{j0} + k_m r_{j,-1} X + k_m r_{j,-2} X^2 + \cdots + k_m r_{j,-m+2} X^{m-2} + k_m r_{j,-m+1} X^{m-1} \\ & + k_{m-1} r_{j0} X + k_{m-1} r_{j,-1} X^2 + \cdots + k_{m-1} r_{j,-m+3} X^{m-2} + k_{m-1} r_{j,-m+2} X^{m-1} \\ & + k_{m-2} r_{j0} X^2 + \cdots + k_{m-2} r_{j,-m+4} X^{m-2} + k_{m-2} r_{j,-m+3} X^{m-1} \\ & \qquad \vdots \qquad\qquad \vdots \\ & + k_2 r_{j0} X^{m-2} + k_2 r_{j,-1} X^{m-1} \\ & + k_1 r_{j0} X^{m-1} \end{aligned}$$
$$\begin{aligned} = {} & k_m r_{j0} - r_{j,m-1} X - r_{j,m-2} X^2 - \cdots - r_{j2} X^{m-2} - r_{j1} X^{m-1} \\ & - k_1 r_{j,m-2} X - k_1 r_{j,m-3} X^2 - \cdots - k_1 r_{j1} X^{m-2} \\ & \qquad \vdots \qquad\qquad \vdots \\ & - k_{m-3} r_{j2} X - k_{m-3} r_{j1} X^2 \\ & - k_{m-2} r_{j1} X \end{aligned}$$
$$= K_j(X),$$

where the polynomial $K_j(X)$ is defined by:

$$K_j(X) = \begin{cases} k_m & \text{if } j = 1, \\ -k_{m-j}X - k_{m-j-1}X^2 - \cdots - k_0 X^{m-j+1} & \text{if } 2 \leqslant j \leqslant m. \end{cases} \tag{2.2}$$

Thus,

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = \frac{K_j(X)}{k_m(1 - \alpha_1^{-1}X)(1 - \alpha_2^{-1}X) \ldots (1 - \alpha_m^{-1}X)}.$$

For $i = 1, \ldots, m$, let k_i^* denote the i^{th} elementary symmetric functions in $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$, so that

$$k_i^* = (-1)^i k_{m-i}/k_m,$$

as is well known. Now, let p_t^* stand for the sum of all symmetric functions of weight t in $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$; letting

$$\begin{aligned} F(Y) &= Y^m + \frac{k_{m-1}}{k_m} Y^{m-1} + \cdots + \frac{k_1}{k_m} Y + \frac{1}{k_m} \\ &= (Y - \alpha_1^{-1})(Y - \alpha_2^{-1}) \ldots (Y - \alpha_m^{-1}), \end{aligned}$$

we have

$$p_t^* = \sum_{i=1}^{m} \frac{(\alpha_i^{-1})^{m-1+t}}{F'(\alpha_i^{-1})},$$

and $p_{-1}^* = p_{-2}^* = \cdots = p_{-m+1}^* = 0$.

We conclude that

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = k_m^{-1} K_j(X)(1 + p_1^* X + p_2^* X^2 + \cdots + p_t^* X^t + \cdots),$$

and this leads to

$$r_{j,-n} = \begin{cases} p_n^* & \text{if } j = 1, \\ -\dfrac{1}{k_m} \displaystyle\sum_{t=0}^{m-j} k_{m-j-t} p_{n-1-t}^* & \text{if } j = 2, \ldots, m. \end{cases} \tag{2.3}$$

Instead of using p_t (resp. p_t^*), one can also use the multinomial theorem from [7] to find r_{jn} (resp. $r_{j,-n}$). For example, as in [10], we have (within an irrelevant radius of convergence)

$$\begin{aligned} &\sum_{n=0}^{\infty} r_{jn} X^n \\ &= (X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}) \left[\sum_{j=0}^{\infty} (-1)^j (k_1 X + \cdots + k_{m-1} X^{m-1} + k_m X^m)^j \right] \\ &= (X^{j-1} + k_1 X^j + \cdots + k_{m-j} X^{m-1}) \left(\sum_{i=0}^{\infty} A(i) X^i \right), \end{aligned}$$

where

$$A(i) = \sum (-1)^{t_1 + t_2 + \cdots + t_m} \frac{(t_1 + t_2 + \cdots + t_m)!}{t_1! t_2! \ldots t_m!} k_1^{t_1} k_2^{t_2} \ldots k_m^{t_m},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ of $\mathbb{N}^m$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i;$$

therefore,

$$r_{jn} = \sum_{t=0}^{m-j} k_t A(n - j + 1 - t), \tag{2.4}$$

with the convention $k_0 = 1$, and $A(i) = 0$ for i a negative integer.

Similarly, for $j = 2, \ldots, m$, we have

$$\sum_{n=0}^{\infty} r_{j,-n} X^n = (-k_{m-j}X - k_{m-j-1}X^2 - \cdots - k_0 X^{m-j+1})\left(\sum_{i=0}^{\infty} B(i)X^i\right),$$

where

$$B(i) = \sum \frac{(-1)^{t_1+t_2+\cdots+t_m}(t_1 + t_2 + \cdots + t_m)!}{k_m^{t_1+t_2+\cdots+t_m+1} t_1! t_2! \ldots t_m!} k_{m-1}^{t_1} k_{m-2}^{t_2} \ldots k_1^{t_{m-1}},$$

the last sum being taken over all m-tuples $(t_1, t_2, \ldots, t_m)$ such that

$$t_1 + 2t_2 + \cdots + mt_m = i.$$

Defining $B(i)$ to be 0 for $i < 0$, we therefore obtain

$$r_{j,-n} = \begin{cases} k_m B(n) & \text{if } j = 1, \\ -\sum_{t=0}^{m-j} k_{m-j-t} B(n - 1 - t) & \text{if } j = 2, \ldots, m. \end{cases} \tag{2.5}$$

Although formulas (2.4) and (2.5) with $j = 1$ may look different from Bernstein's formulas (1.14) and (1.14a) in [3], they are in fact equivalent.

To conclude this section, let us remark that if one wants a formula for the powers α^n for $\alpha = a_1 + a_2\omega + \cdots + a_m\omega^{m-1}$, one can use the characteristic polynomial of α to write an equation of the form

$$\alpha^m + b_1\alpha^{m-1} + \cdots + b_m = 0,$$

and apply the above procedure to get the powers of α as functions of the b_i's.

3. A GENERAL RESULT

If $\alpha = \sum_{i=1}^{m} a_{i1}\omega^{i-1} \in \mathbb{Q}(\omega)$ and if, for $j = 1, \ldots, m$,

$$\alpha\omega^{j-1} = \sum_{i=1}^{m} a_{ij}\omega^{i-1},$$

then $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\alpha) = \det A$, where $A = [a_{ij}]$; see [11].

Let us consider the equality

$$\gamma = \alpha\beta$$

with $\beta \in \mathbb{Q}(\omega)$ where, for $j = 1, \ldots, m$,

$$\beta\omega^{j-1} = \sum_{i=1}^{m} b_{ij}\omega^{i-1}, \quad \gamma\omega^{j-1} = \sum_{i=1}^{m} g_{ij}\omega^{i-1};$$

taking $B = [b_{ij}]$, $G = [g_{ij}]$, $\Omega = [1 \quad \omega \quad \dots \quad \omega^{m-1}]$, we have

$$\alpha\Omega = \Omega A, \ \beta\Omega = \Omega B, \ \gamma\Omega = \Omega G \quad \text{and} \quad (\alpha\beta)\Omega = (\beta\alpha)\Omega = \Omega(BA) = \Omega(AB),$$

hence, the identity

$$G = AB = BA.$$

If, for a matrix M, we denote its j^{th} column by $M_{(j)}$, we conclude

$$\alpha\beta = \gamma = \Omega G_{(1)} = \Omega AB_{(1)} = \Omega BA_{(1)}.$$

In particular, we obtain $AB_{(1)} = G_{(1)}$, i.e.,

$$\begin{cases} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1} = g_{11} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2m}b_{m1} = g_{21} \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mm}b_{m1} = g_{m1}. \end{cases}$$

Let $\alpha \neq 0$; then $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\alpha) = \det A \neq 0$, and the matrix A of the coefficients of the above system has $\det A \neq 0$. For $i = 1, \dots, m$, Cramer's rule gives

$$b_{i1} = \frac{1}{\det A}\sum_{t=1}^{m} g_{t1}\operatorname{cof}(a_{ti}), \tag{3.1}$$

where $\operatorname{cof}(a_{ti})$ is the cofactor of the (t, i) element of A.

Similarly, if $\beta \neq 0$, we also have, for $i = 1, \dots, m$,

$$a_{i1} = \frac{1}{\det B}\sum_{t=1}^{m} g_{t1}\operatorname{cof}(b_{ti}). \tag{3.2}$$

In [2] Bernstein took $m = 3$, $k_1 = 0$, $k_2 = g \geqslant 2$, $k_3 = -1$, $\alpha = \omega^s$, $\beta = \omega^{-s}$, $\gamma = 1$, obtained recurrence relations for the rational coefficients of α and β, calculated the generating function of these coefficients, and then obtained his combinatorial identities, which turn out to be special cases of our formulas (3.1) and (3.2); see formulas (4.2) and (4.3) of [2], see also [1], [4], and [5]. The same was done in a lengthy way for arbitrary m in [3]. It turns out that in [3], Bernstein is considering $\alpha = \omega^{m-n+1}$, $\beta = \alpha^{-1}$, $\gamma = 1$; nevertheless, he forgot to write a_0 in front of the determinant appearing in his formula (2.3b), so formulas (2.4)-(2.7) must be modified accordingly [e.g., the power of a_0 in (2.7) is m].

Let us observe that, from a linear algebra point of view, the equality $G = AB$ with $\det A \neq 0$ immediately implies that one can solve for the entries of B in terms of the entries of G and minors of A.

4. VANDERMONDE MATRIX TREATMENT

Consider the matrix C defined in Section 1, the Vandermonde matrix V, and the diagonal matrix D shown on the following page:

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{m-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{m-1} \end{bmatrix}, \qquad D = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \alpha_m \end{bmatrix}.$$

Since $VC = DV$, we have $C = V^{-1}DV$, and

$$\det V = |V| = \prod_{j>i} (\alpha_j - \alpha_i).$$

It is possible to give an explicit formula for C in terms of det V and in terms of the determinant of a certain matrix that is almost of the Vandermonde type. Let us do it.

For $t = 1, \dots, m - 1$, denote by $k_t(j)$ the t^{th} elementary symmetric function in $\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_m$, whence

$$k_t(j) = k_t + k_{t-1}\alpha_j + \cdots + k_1\alpha_j^{t-1} + \alpha_j^t. \tag{4.1}$$

With respect to V, define

$$V_i = \text{cof}(\alpha_i^{m-1}).$$

Then it is well known that

$$V^{-1} = \frac{1}{|V|}\begin{bmatrix} k_{m-1}(1)V_1 & k_{m-1}(2)V_2 & \dots & k_{m-1}(m)V_m \\ k_{m-2}(1)V_1 & k_{m-2}(2)V_2 & \dots & k_{m-2}(m)V_m \\ \vdots & \vdots & & \vdots \\ k_1(1)V_1 & k_1(2)V_2 & \dots & k_1(m)V_m \\ V_1 & V_2 & \dots & V_m \end{bmatrix}$$

(see for instance [9]). For a proof, call W the matrix $|V|V^{-1}$, and show that $WV = |V|I_m$ by comparing the (i, j) elements: if $i = j$, you obtain $|V|$; if $j < i$, you get 0, using (4.1); if $j > i$, you obtain 0, using the fact that

$$\alpha_t^{j-1}k_{m-i}(t) = -k_{m-i+1}\alpha_t^{j-2} - k_{m-i+2}\alpha_t^{j-3} - \cdots - k_{m-1}\alpha_t^{j-i} - k_m\alpha^{j-i-1}.$$

By definition, for all $n \in \mathbb{Z}$, let H_n be given by

$$H_n = \det \begin{bmatrix} 1 & \alpha_1 & \dots & \alpha_1^{m-2} & \alpha_1^{n-1} \\ 1 & \alpha_2 & \dots & \alpha_2^{m-2} & \alpha_2^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \alpha_m & \dots & \alpha_m^{m-2} & \alpha_m^{n-1} \end{bmatrix};$$

as is easily verified, this determinant H_n satisfies the m^{th}-order linear recurrence

$$H_n = -k_1H_{n-1} - k_2H_{n-2} - \cdots - k_mH_{n-m}.$$

Keeping in mind formula (4.1), we find for the (j, t) element of $C^n = V^{-1}D^nV$ (with $1 \leqslant j, t \leqslant m$):

$$(j, t) \text{ element of } C^n = \frac{1}{|V|}\sum_{i=0}^{m-j} k_{m-j-i}H_{n+t+i}.$$

So, for every $n \in \mathbb{Z}$,

$$r_{jn} = \frac{1}{|V|}\sum_{i=0}^{m-j} k_{m-j-i}H_{n+1+i}. \tag{4.2}$$

REFERENCES

1. L. Bernstein. "Zeros of the Functions $f(n) = \sum_{i=0}^{\infty} (-1)^i \binom{n-2i}{i}$." *J. Number Theory 6*, no. 4 (1974):264-270.
2. L. Bernstein. "Zeros of Combinatorial Functions and Combinatorial Identities." *Houston J. Math. 2*, no. 1 (1976):9-16.
3. L. Bernstein. "Periodic Algorithms and Their Application." *Can. J. Math. 31*, no. 3 (1979):565-595.
4. L. Bernstein. "Applications of Units." *J. Number Theory 10* (1978):354-383.
5. L. Bernstein. "An Invariant for Combinatorial Identities." *The Fibonacci Quarterly 16*, no. 4 (1978):354-369.
6. W. S. Burnside. *The Theory of Equations*. 7th ed. 2 vols. New York: Dover Publications.
7. G. Chrystal. *Algebra*. 6th ed. Vol. II. New York: Chelsea, 1952.
8. B. N. Delone & D. K. Faddeev. *The Theory of Irrationalities of the Third Degree*. AMS Translations, Vol. 10, 1964.
9. A. Klinger. "The Vandermonde Matrix." *American Math. Monthly 74*, no. 5 (1967):571-574.
10. C. Levesque. "On m^{th} Order Linear Recurrences." *The Fibonacci Quarterly 23*, no. 4 (1985):290-293.
11. P. Samuel. *Théorie algébrique des nombres*. Paris: Herman, 1967.

◆◇◆◇◆