LUCAS SEQUENCES IN SUBGRAPH COUNTS OF SERIES-PARALLEL AND RELATED GRAPHS

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1. INTRODUCTION

We follow graph theoretic terminology as in [B&M]. Let $G = (V, E)$ denote a graph where V is a set of vertices and E is a set of nonoriented edges. Though we do not in general consider graphs with loops or multiple edges, we make reference to such graphs for the purpose of proofs. When an edge e appears m times, we say e has *multiplicity* m. A *subgraph* $G' = (V', E')$ of G is any graph such that $V' \subseteq V$ and $E' \subseteq E$, and a *spanning subgraph* of G contains every vertex of V. A sequence of vertices $n_1, n_2, n_3, \ldots, n_k$ is a *path* of G if $n_i \in V$, $\{n_i, n_{i+1}\} \in E$, for all i, and no vertices are repeated. A path is a *cycle* if $n_1 = n_k$. A *tree* of G is a subgraph with no cycles; a *spanning tree* contains every vertex of G. Let $r(G)$ denote the count of spanning trees of G.

Spanning tree counts of general graphs can be obtained in $O(n^3)$ time by computing the determinant of its in-degree matrix [7], where n is the number of vertices. This function grows quickly; as well, the practical interest of circuit theory in counting spanning trees motivates the study of classes of graphs for which spanning tree counts can be obtained in linear time.

Sedlacek [19] notes that W_{n+1}, the *wheel* on $n + 1$ vertices, is obtained from a cycle on n points we call the *rim* by joining each point in the cycle to another point we call the *hub*. Vertices and edges on the rim are *rim vertices* and *rim edges*; an edge joining a rim vertex and the hub is a *spoke*. Sedlacek considers F_{n+1}, the *auxiliary fan* of W_{n+1}, derived from W_{n+1} by removing a single rim edge and proves

$$r(F_{n+1}) = \frac{(3+\sqrt{5})^{n+1} - (3-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}$$

and

$$r(W_{n+1}) = \left(\frac{3+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} - 2.$$

It is remarkable that $r(F_{n+1})$ generates every second number of the Fibonacci series.

Myers [14] and Bedrosian [2] derive similar formulas for wheels and multigraph wheels in a circuit theory setting. Hilton [10] presents formulas for $r(G)$ of fans and wheels in terms of Fibonacci and Lucas numbers, and Fielder [8] provides tree counts for *sector graphs*, fans with certain multiple edges. Slater [21] shows that all maximal outerplane graphs with exactly two vertices of degree two have the same spanning tree count as fans. (We coin the term *generalized fan* to refer to these graphs in [16] and [17].) Shannon [20] derives $r(W_{n+1})$ with a number theoretic approach. Bange, Barkauskas, and Slater

[1] show that generalized fans have more spanning trees than any other maximal outerplane graph. Most of these studies have been motivated by the remarkable involvement of Fibonacci numbers in spanning tree counts.

The study of *network reliability* demands counts of subgraphs other than spanning trees. Previously, formulas for subgraph counts apparently existed only for complete graphs [9]. A network is commonly modeled as a *probabilistic graph* where each edge e fails independently with probability p and vertices never fail. The probability that such a graph is connected is called *probabilistic connectedness*, and is a standard measure of network reliability. This can be generalized in two different ways. In some applications, a network may not be considered operational unless it has edge connectivity or *cohesion* of at least k; this we call k-cohesive connectedness. Alternately, a network may be considered operational if it has broken down into no more than k components; we call this k-component connectedness. In Section 2, we use Lucas recurrences to count various types of subgraphs of generalized fans and related graphs. Section 3 counts connected spanning subgraphs with cohesion of at least two (two-cohesive). Section 4 presents the rank polynomial as a technique for classifying subgraphs of generalized fans both by number of edges and by number of components. We conclude in Section 5 with some applications. By noting that probabilities can be encoded in the coefficients of some of these recurrences, we obtain reliability formulas as well as subgraph counting formulas. As in previous studies, we find that the required enumerations are given in two-term recurrence relations; hence, the desired subgraph counts are Lucas numbers.

2. COUNTING CONNECTED SPANNING SUBGRAPHS

We begin by counting connected spanning subgraphs of generalized fans that satisfy a Lucas recurrence. Generalized fans are a subset of *2-trees* [18], defined recursively as follows:

1) A single edge is a 2-tree.
2) If G is a 2-tree with edge $\{x, y\}$, adding a new vertex z, and the two edges $\{x, z\}$ and $\{y, z\}$ creates a new 2-tree. If G is not a single edge, $\{x, y\}$ becomes an *interior edge* of the new graph.

When parallel edges are not allowed, 2-trees are equivalent to *maximal series-parallel networks* as in [6], [16], [17]; other definitions of series-parallel networks do appear in the literature.

Any vertex of degree two is a *peripheral vertex*; an edge incident on a peripheral vertex is a *peripheral edge*. To illustrate the counting technique, we reproduce in part this lemma from [16] which counts connected spanning subgraphs of generalized fans.

Lemma 2.1: The number of connected spanning subgraphs of an n-vertex generalized fan, $sc(n)$, satisfies the recurrence:

$$sc(n) = 4sc(n - 1) - 2sc(n - 2).$$

Proof: Let **peripheral** vertex z be attached to edge $\{x, y\}$ of generalized fan G by edges $\{x, z\}$ and $\{y, z\}$. A connected spanning subgraph of G induces on $G - z$ either a connected spanning subgraph or a disconnected spanning subgraph which the addition of $\{x, y\}$ would connect. To handle this latter case, we define $dc(n)$ to be the number of spanning subgraphs of an n-vertex generalized fan which the addition of a specific peripheral edge would connect.

Any connected spanning subgraph of G must contain at least one of $\{x, z\}$ and $\{y, z\}$. If both are selected, the graph induced on $G - z$ must either be connected, or be one of the graphs counted by dc. In this case, there are $sc(n-1) + dc(n-1)$ induced subgraphs. Otherwise, a connected spanning subgraph contains either $\{x, z\}$ or $\{x, z\}$, but not both. But then the graph induced on $G - z$ must be connected; the number in this case is $2sc(n-1)$. Therefore,

$$sc(n) = 3sc(n-1) + dc(n-1).$$

By a similar argument,

$$dc(n) = sc(n-1) + dc(n-1).$$

These two recurrences may be combined to yield

$$sc(n) = 4sc(n-1) - 2sc(n-2). \blacksquare$$

Since $sc(2) = 1$ and $sc(3) = 4$, the recurrence yields the closed formula

$$sc(n) = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{2\sqrt{2}}.$$

From a reliability perspective, it is interesting that all generalized fans have the same number of connected spanning subgraphs; in addition, generalized fans have more connected spanning subgraphs than any other 2-tree [16]. We say F_i is a *subfan* of the fan F_n if h, the hub of F_n, is a vertex in F_i, F_i is a subgraph of F_n and F_i is a fan. From Lemma 2.1, we then show:

Lemma 2.2: For $n \geq 4$, the number of connected spanning subgraphs of a wheel on n vertices, $sc_W(n)$, is

$$sc_W(n) = 2\sum_{i=2}^{n} sc(i).$$

Proof: Consider the n-vertex wheel W_n with rim edge $\{a, b\}$. Denote by F_n the auxiliary fan of W_n created by removing $\{a, b\}$.

A connected spanning subgraph of W may or may not contain $\{a, b\}$. If not, there are $sc(n)$ connected spanning subgraphs of the auxiliary fan of W_n which are also connected spanning subgraphs of W_n. But we can also add the edge $\{a, b\}$ to any of the connected spanning subgraphs of F_n and get a connected spanning subgraph of W_n.

Lastly, the edge $\{a, b\}$ connects any *two-component* spanning subgraph of F_n, one containing a and the other containing b. Such a spanning subgraph of F_n must consist of a path on $n - i$ vertices and a connected spanning subgraph of the subfan of F_n on the remaining i vertices.

For each i, there are exactly two ways we can choose a path on $n - i$ vertices containing exactly one of a or b, and $sc(i)$ ways of obtaining a connected spanning subgraph on the remaining i vertices; hence, for each i we obtain $2sc(i)$ connected spanning subgraphs of W_n. We vary i from 2 to $n - 1$, and the result follows. ∎

The above simplifies to:

$$sc_W(n) = (2+\sqrt{2})^{n-1} + (2-\sqrt{2})^{n-1} - 2.$$

This is analogous to Sedlacek's formula for spanning trees in a wheel.

3. COUNTING 2-COHESIVE SPANNING SUBGRAPHS

Sometimes a network must be at least k-cohesive, i.e., the order of the minimum edge cutset must be at least k, to be operational. This happens in an environoment where queuing delay is a problem [26]. The number of 2-cohesive spanning subgraphs of generalized fans satisfies a recurrence of the Lucas type [15]. We state the following without proof.

Lemma 3.1: For $n \geqslant 3$, $sc_2(n)$, the number of 2-cohesive spanning subgraphs of an n-vertex generalized fan is

$$sc_2(n) = dc_2(n) = 2sc_2(n-1) + sc_2(n-2)$$

$$= \frac{\sqrt{2}}{4}[(1+\sqrt{2})^{n-2} - (1-\sqrt{2})^{n-2}]. \blacksquare$$

As before, the count of two-connected spanning subgraphs is maximized by minimizing the number of peripheral vertices.

4. THE RANK POLYNOMIAL OF A GENERALIZED FAN

Subgraph counts have been studied in an algebraic setting by Tutte [22], [23], [24], and [25] and others [3] and [5]. In this section, we derive the *rank polynomial* of a generalized fan, by a similar technique.

Let $c(G)$ denote the number of components of a graph G. In addition, write

$$i(G) = |V| - c(G), \qquad j(G) = |E| - |V| + c(G).$$

If S is any subset of E, $G{:}S$ denotes the subgraph of G induced by S. Then denote by $RK(G;\, t,\, z)$ the rank polynomial of G where

$$RK(G;\, t,\, z) = \sum_{S \subseteq E} t^{i(G:S)} z^{j(G:S)}.$$

Note that $i(G{:}S) + j(G{:}S) = |S|$; thus, from the rank polynomial of a graph, we can quickly classify spanning subgraphs of G not only by number of edges but also by number of components. From [24], we can trivially derive the following three properties of the rank polynomial which completely characterize $RK(G;\, t,\, z)$:

1) If G consists of two vertex disjoint subgraphs H and K, then

$$RK(G;\, t,\, z) = RK(H;\, t,\, z)RK(K;\, t,\, z).$$

2) (Rank polynomial factoring theorem). If e is any edge in E,

$$RK(G;\, t,\, z) = RK(G - e;\, t,\, z) + tRK(G \bullet e;\, t,\, z),$$

where $G \bullet e$ is graph G less edge $e = \{x, y\}$ with endvertices x and y identified.

3) If G consists of a single vertex and k loops,

$$RK(G;\, t,\, z) = (1+z)^k.$$

Thus, the rank polynomial is a rich source of information about subgraph counts. We need some more identities:

Lemma 4.1: (a) If G is a single edge on two vertices, then

$$RK(G;\ t,\ z) = 1 + t.$$

(b) If G_x is the graph derived by adding a loop to any vertex x of G, then

$$RK(G_x;\ t,\ z) = (1 + z)RK(G;\ t,\ z).$$

(c) If $G = H \cup K$ and $H \cap K$ contains no edges and exactly one vertex,

$$RK(G;\ t,\ z) = RK(H;\ t,\ z)RK(K;\ t,\ z).$$

Proof: (a) Note that if H is any edgeless graph, then $RK(H;\ t,\ z) = 1$. A single application of the rank polynomial factoring theorem yields the result. ■

(b) Any spanning subgraph of G is a spanning subgraph of G_x; to each spanning subgraph of G, we can add the edge $\{x,\ x\}$ also yielding another spanning subgraph of G_x. This second set of spanning subgraphs can be represented by multiplying the rank polynomial of G by z, i.e., increasing the edge count of every term in the polynomial without disturbing any other information. ■

(c) Consider any subgraphs H' and K' of H and K, respectively. H' has n_H vertices, e_H edges, and c_H components. Similarly, K' has n_K vertices, e_K edges, and c_K components. The subgraph $H' \cup K'$ of G has $n_H + n_K - 1$ vertices, $e_H + e_K$ edges, and $c_H + c_K - 1$ components. Expressing the term of $RK(G;\ t,\ z)$ corresponding to $H' \cup K'$ in terms of the corresponding expressions for H' and K' in $RK(H;\ t,\ z)$ and $RK(K;\ t,\ z)$ yields the desired result. ■

We have seen that every generalized fan on n vertices, regardless of topology, has the same number of connected spanning subgraphs. Nevertheless, it is surprising that all n-vertex generalized fans have the same rank polynomial, again satisfying a two-term linear Lucas recurrence.

Lemma 4.2: The rank polynomial of any generalized fan on n vertices, $S(n)$, satisfies the recurrence

$$S(n) = (1 + 3t + tz)S(n - 1) - t(1 + t)(1 + z)S(n - 2)$$

which may be solved for the closed formula

$$S(n) = \frac{1 + 2t + 3t^2 + t^2z - tz + (1 + t)\alpha}{2\alpha}\left(\frac{1 + 3t + tz + \alpha}{2}\right)^{n-2} - \frac{1 + 2t + 3t^2 + t^2z - tz - (1 + t)\alpha}{2\alpha}\left(\frac{1 + 3t + tz - \alpha}{2}\right)^{n-2}$$

where $\alpha = \sqrt{(1 + 3t + tz)^2 + 4t(1 + t)(1 + z)}$.

Proof: As preliminaries, consider some special cases. Let $H_n = G_{n-1} \cup \{x,\ y\}$, be an n-vertex graph where G_{n-1} is an $n-1$-vertex generalized fan with peripheral vertex x and y is a new vertex not in G_{n-1}; then,

$$RK(H_n;\ t,\ z) = (1 + t)S(n - 1)$$

by Lemma 4.1(a) and (c). Let G be an n-vertex generalized fan and write $DD(n)$

for the rank polynomial of an n-vertex generalized fan with one peripheral edge of multiplicity 2. A single application of the rank polynomial factoring theorem to one of G's peripheral edges yields

$$S(n) = RK(H_n;\ t,\ z) + DD(n-1) = (1+t)S(n-1) + DD(n-1).$$

We obtain a recursive expression for $DD(n)$ by applying the rank polynomial factoring theorem to one of the edges of multiplicity 2. If G is an n-vertex generalized fan with one peripheral edge e of multiplicity 2, then $G - e$ is an n-vertex generalized fan and $G \bullet e$ is an $n-1$-vertex generalized fan with a peripheral edge of multiplicity 2 and a loop at the peripheral vertex. Then

$$DD(n) = S(n) + t(1+z)DD(n-1)$$

by Lemma 4.1(b).

Combining these expressions provides the stated two-term Lucas recurrence, and solving gives the closed formula. ■

5. APPLICATIONS

Subgraph counts alone provide a measure of the connectedness of a graph. However, the recurrences in Section 2 can be generalized to compute probabilistic connectedness or, alternately, two-cohesive connectedness. If p is the probability that a single edge is up, then $R_p(n)$ is the probability that an n-vertex generalized fan is connected. Let $\rho_p(n)$ be the probability of obtaining a spanning subgraph on n vertices that *would become* connected if a *specific* peripheral edge were added. Since the context is clear, we omit the probability subscript. The following is a new proof of the main result in [17] using Lucas recurrences rather than generating functions.

Theorem 5.1: Let $x = q/p$. $R(n)$, the probability that an n-vertex generalized fan is connected is given by:

$$R(n) = p^2(3x+1)R(n-1) - p^4(x^2+x)R(n-2).$$

It is remarkable that $\rho(n)$ also obeys the same relation, that is:

$$\rho(n) = p^2(3x+1)\rho(n-1) - p^4(x^2+x)\rho(n-2).$$

Proof: Consider the n-vertex generalized fan G having peripheral vertex z and edge of attachment $\{x, y\}$. We measure $R(n)$ as a product of the states of edges $\{x, z\}$, $\{y, z\}$ and the subgraph induced by $G - z$. The probability that $G - z$ is connected is $R(n-1)$; the probability that at least one of $\{x, z\}$ and $\{y, z\}$ is up is $2pq + p^2$. The probability of a connected spanning subgraph in this case is $R(n-1)(2pq + p^2)$. Suppose, on the other hand, $G - z$ is disconnected but the addition of $\{x, y\}$ would connect it; if both $\{x, z\}$ and $\{y, z\}$ are up, the resultant subgraph of G is connected with probability $p^2\rho(n-1)$. Then

$$R(n) = p^2\rho(n-1) + (2pq + p^2)R(n-1).$$

Similarly,

$$\rho(n) = pq\rho(n-1) + q^2R(n-1).$$

Combining these formulas yields the stated recurrences which can then be solved for a closed formula for probabilistic connectedness. ■

Such formulas are extremely useful, since it appears that no other exact measures of probabilistic connectedness exist except for complete graphs [9].

A formula for 2-cohesive connectedness can be derived similarly; in both instances, generalized fans are the most reliable maximal series-parallel network (see [15] and [17]).

The rank polynomial of a generalized fan yields a family of reliability measures. Let $t = z/r$ and $z = p/q$, and let $KC(n, k)$ be the probability of obtaining a subgraph of no more than k components. We multiply the rank polynomial by $r^{|V|}q^{2n-3}$ and collect terms by superscripts of z to yield

$$q^{2n-3} \sum_d c_d r^{|V|-i}\left(\frac{p}{q}\right) i^{i+j}.$$

From this, we can write

$$KC(n, k) = q^{2n-3} \sum_d c_d T(|V| - i \leqslant k)\left(\frac{p}{q}\right)^{i+j},$$

where $T(expression)$ returns 1 if its argument is true and 0 otherwise.

Lastly, these techniques apply to other classes of graphs. Generalizing Sedlacek [19], Mikola [13] describes $V_n^{(k)}$ as the path $v_0 v_1 v_2 \cdots v_{(n-1)(k-1)}$ and the edges wv_i for $i = 0, k + 1, 2(k + 1), \ldots, (n - 1)(k - 1)$, i.e., rim edges are replaced with paths of equal length. Then

$$r(V_n^{(k)}) = \frac{((k + 3 + K)^n - (k + 3 - K)^n)}{(2^n K)},$$

where $K = \sqrt{k^2 + 6k + 5}$. We generalize Mikola's result by replacing spokes with paths of equal length. Furthermore, a *generalized Mikola fan* is obtained from a generalized fan by replacing all the interior edges and any two nonadjacent peripheral edges by paths of length $j + 2$ and all the other edges by paths of length $k + 2$.

The connected spanning subgraph count of a generalized Mikola fan, $G(n)$, satisfies the recurrence:

$$G(n) = (k + 2j + 4)G(n - 1) - (j^2 + 3j + 2)G(n - 2),$$

where n is the index as in the definition. Solving this yields a formula for subgraph counts of yet another class of uniformly sparse graphs.

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