A TWO-DIMENSIONAL GENERALIZATION OF GRUNDY'S GAME

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1. INTRODUCTION

A positional game with normal winning rule for two alternately moving persons may be defined by a pair $\Gamma = (P, S)$. P is the finite set of game positions and S is a mapping $S : P \to 2^P$ such that $y \in S(x)$ if and only if y is a position that can be brought about from position x by a legal move. We call $S(x)$ the set of successors of x. A play of Γ terminates when a position x with $S(x) = \emptyset$ has been reached. The normal winning rule states that the player loses who is first unable to move. It is assumed that there is an upper bound for the number of moves in any play. A disjunctive combination of a finite set of such games $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ may be defined thus: The players play alternately, each in turn making a move in one of the individual games. A player loses if unable to move.

P. M. Grundy [1] showed that for $\Gamma = (P, S)$ the function $G : P \to \mathbb{N}_0$ with

$$G(x) = \begin{cases} 0 & \text{if } S(x) = \emptyset \\ \min(\mathbb{N}_0 \setminus \{G(y) \mid y \in S(x)\}) & \text{if } S(x) \neq \emptyset \end{cases} \qquad (*)$$

has the properties

*P*1: A player who moves from a position x with $G(x) > 0$ can consistently move to a positive the G-value of which is 0 and so win the play.

*P*2: If Γ is a disjunctive combination of games $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, then for a combined position $x = (x_1, x_2, \ldots, x_k)$ the G-value is the nim-sum of the individual G-values, i.e.,

$G(x) = G(x_1) \overset{*}{+} G(x_2) \overset{*}{+} \cdots \overset{*}{+} G(x_k)$, where the nim-sum $a \overset{*}{+} b$ for $a, b \in \mathbb{N}_0$ is defined by

$$a \overset{*}{+} b = \begin{cases} 0 & \text{if } a = b = 0, \\ (a + b) \bmod 2 + 2(a \text{ div } 2 \overset{*}{+} b \text{ div } 2) & \text{otherwise,} \end{cases}$$

where a div 2 is the integer division of a by 2.

So the G-value of $x = (x_1, x_2, \ldots, x_k)$ can easily be calculated if $G(x_i)$ is known for $i = 1, 2, \ldots, k$.

2. GRUNDY'S GAME

P. M. Grundy himself invented the following game: The starting position is a single heap of K matches. The first player, by his move, divides this heap into two heaps of unequal size. the second player selects one of the heaps and divides it into two heaps of unequal size, and so forth. The play terminates when each individual heap contains only one or two matches. The second player makes the first move in a game that is the disjunctive combination of the two Grundy-games determined by the two heaps resulting from the first player's move. If the starting position of a Grundy-game is described by the number K of corresponding matches, we get

$$S(K) = \{(I, K - I) \mid 0 < I < K \text{ and } I \neq K - I\}.$$

The G-value of K according to the recursive definition (*) taking into consideration $P2$ is determined by

$$G(K) = \begin{cases} 0 & \text{if } K = 1 \text{ or } K = 2, \\ \min(\mathbb{N}_0 \setminus \{G(I) \overset{*}{+} G(K - I) \mid 0 < I < K,\ I \neq K - I\}) & \text{otherwise.} \end{cases}$$

For $K \leqslant 100$, the following G-values are obtained:

Table 1

K	$G(K)$			
1- 40	0010210210	2132132430	4304304123	1241241241
41- 80	5415415410	2102152132	1324324324	3243243245
81-100	2452437437	4374352352		

(See, also [4].)

3. THE GENERALIZED GRUNDY-GAME

A play in this game starts with an $(M \times N)$-rectangle. The first player breaks this rectangle into two rectangles of unequal size, to that the sides of the resulting rectangles are of integer length. (Imagine a bar of chocolate that can be broken along vertical and horizontal scores.) The second player breaks one of these rectangles into two new rectangles of unequal size, and so on. The play terminates when both sides of each individual rectangle are less than or equal to 2. According to the normal winning rule, it is lost by the person who is first unable to move.

$G(M, N)$ denotes the G-value corresponding to a single $(M \times N)$-rectangle, while $G(K)$ refers to a position in Grundy's original game. The following properties are obvious:

$Q1$: $G(M, N) = G(N, M)$, and

$Q2$: $G(1, N) = G(2, N) = \mathbf{G}(N)$.

(The generalized game starting with a $(1 \times N)$-rectangle or a $(2 \times N)$-rectangle is obviously equivalent to the original Grundy-game starting with N matches.)

The set of positions succeeding to (M, N) is:

$$S(M, N) = \begin{cases} \emptyset \quad \text{if } M \leqslant 2 \text{ and } N \leqslant 2, \\ \{((m, N), (M - m, N)) \mid 0 < m < M \text{ and } m \neq M - m\} \\ \cup \{((M, n), (M, N - n)) \mid 0 < n < N \text{ and } n \neq N - n\} \quad \text{otherwise.} \end{cases}$$

So $G(M, N)$ can be calculated according to:

$$G(M,\ N) = \begin{cases} 0 \quad \text{if } M \leqslant 2 \text{ and } N \leqslant 2, \\ \min\left(\mathbb{N} \setminus \left(\{G(m,\ N) \overset{*}{+} G(M-m,\ N) \mid 0 < m < M,\ m \neq M-m\} \right.\right. \\ \left.\left. \cup \{G(M,\ n) \overset{*}{+} G(M,\ N-n) \mid 0 < n < N,\ n \neq N-n\}\right)\right) \text{ otherwise,} \end{cases}$$

resulting in Table 2:

Table 2

M \ N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	...
1	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
2	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	2	0	
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
4	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
5	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
7	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
8	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
10	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
11	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
12	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
13	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
14	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
16	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
17	2	2	1	2	1	1	2	1	1	2	1	1	1	1	1	1	1	1	1	2	
18	4	4	1	4	1	1	4	1	1	4	1	1	1	1	1	1	1	1	1	4	
19	3	3	1	3	1	1	3	1	1	3	1	1	1	1	1	1	1	1	1	3	
20	0	0	1	0	2	1	0	2	1	0	2	1	3	2	1	3	2	4	3	0	
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This table indicates that $G(M,\ N)$ is completely determined by the values $G(M)$ and $G(N)$, as the following theorem states.

Theorem:

$$G(M,\ N) = \begin{cases} G(N) & \text{if } G(M) = 0, \\ G(M) & \text{if } G(N) = 0, \\ 1 & \text{if } G(M) > 0 \text{ and } G(N) > 0, \end{cases}$$

for all $M,\ N \in \mathbb{N}$.

Proof (Induction on $M + N$):

$M + N = 2$: $G(1,\ 1) = 0 = G(1)$,

$M + N = 3$: $G(1,\ 2) = G(2,\ 1) = 0 = G(1) = G(2)$.

Now assume that there is an $(I \times K)$-rectangle such that $I + K = n + 1$, where $n \geqslant 2$, and that the theorem is true for all $(M \times N)$-rectangles with $M + N \leqslant n$.

Under this assumption, we must prove the following:

1. $G(I) = 0 \Rightarrow G(I,\ K) = G(K)$,
2. $G(K) = 0 \Rightarrow G(I,\ K) = G(I)$, and
3. $G(I) > 0$ and $G(K) > 0 \Rightarrow G(I,\ K) = 1$.

1. $G(I) = 0 \Rightarrow \min(\mathbb{N}_0 \backslash \{G(I, k) \overset{*}{+} G(I, K - k) | 0 < k < K, k \neq K - k\})$
$= \min(\mathbb{N}_0 \backslash \{G(k) \overset{*}{+} G(K - k) | 0 < k < K, k \neq K - k\}) = G(K).$

It remains to prove that

$G(K) \notin \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}.$

It is either $G(K) = 0$ or $G(K) > 0$. These two cases will be inspected separately.

(i) $G(K) = 0$:

$\min(\mathbb{N}_0 \backslash \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}) =$

$\min(\mathbb{N}_0 \backslash \{G(i) \overset{*}{+} G(I - i) | 0 < i < I, i \neq I - i\}) = G(I) = 0.$

Since $G(K) = 0$, it is

$G(K) \notin \{G(i, K) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}.$

(ii) $G(K) > 0$:

$\{G(i, K) | 0 < i < I\} \subset \{1, G(K)\}$ as $i + K \leqslant n$.

Because of $1 \overset{*}{+} 1 = 0$,

$G(K) \overset{*}{+} G(K) = 0$, and

$$1 \overset{*}{+} G(K) = \begin{cases} G(K) + 1 & \text{if } G(K) \text{ is even,} \\ G(K) - 1 & \text{if } G(K) \text{ is odd,} \end{cases}$$

yields $G(i, K) \overset{*}{+} G(I - i, K) \neq G(K)$ for $0 < i < I$, and therefore

$G(K) \notin \{G(K, I) \overset{*}{+} G(I - i, K) | 0 < i < I, i \neq I - i\}.$

2. $G(K) = 0 \Rightarrow G(K, I) = G(I)$, as was just proved, and $G(K, I) = G(I, K)$.

3. $G(I) > 0$ and $G(K) > 0$:

In this case, $I > 2$ and $K > 2$.

$G(I, K) = 1$ is equivalent to:

(i) $G(i, K) \overset{*}{+} G(I - i, K) \neq 1$ for all i such that $0 < i < I$ and $i \neq I - i$;

(ii) $G(I, k) \overset{*}{+} G(I, K - k) \neq 1$ for all k such that $0 < k < K$ and $k \neq K - k$;

(iii) $G(i, K) \overset{*}{+} G(I - i, K) = 0$ for some i such that $0 < i < I$ and $i \neq I - i$

or

$G(I, k) \overset{*}{+} G(I, K - k) = 0$ for some k such that $0 < k < K$ and $k \neq K - k$.

To prove (i), (ii), and (iii) of 3:

(i) By assumption, $G(i, K) \in \{1, G(K)\}$ for $0 < i < I$. So we have

$$G(i, K) \overset{*}{+} G(I - i, K) = \begin{cases} 1 \overset{*}{+} 1 = 0 & \text{or} \\ G(K) \overset{*}{+} G(K) = 0 & \text{or} \\ 1 \overset{*}{+} G(K) \neq 1 \end{cases}$$

for all i, $0 < i < I$.

(ii) $G(I, k) + G(I, K - k) = G(k, I) \overset{*}{+} G(K - k, I) \neq 1$ for all k, $0 < k < K$, because of (i).

(iii) Suppose $G(i, K) \overset{*}{+} G(I - i, K) \neq 0$ for a certain i, $0 < i < I$ and $i \neq I - i$. This is equivalent to $G(i, K) \neq G(I - i, K)$. Since, by assumption, $G(i, K)$ is determined by $G(i)$ and $G(K)$, it follows that $G(i) \neq G(I - i)$ for this i.

However, there must exist an i, $0 < i < I$ and $i \neq I - i$ such that $G(i) = G(I - i)$ because otherwise $G(i) \overset{*}{+} G(I - i) \neq 0$ for all i, $0 < i < I$ and $i \neq I - i$. That would mean $G(I) = 0$, in contradiction to the assumption $G(I) > 0$.

4. FURTHER GENERALIZATIONS

The theorem can be extended in different directions:

1. Consider the game in which a heap of K matches may be split into two nonempty heaps of size I and $K - I$ provided $|K - I| \geqslant d$, $d \in N_0$. (For $d = 1$, it is Grundy's original game.)

   For this game and its two-dimensional analogue, the above theorem remains valid. The proof requires only replacing the conditions $i \neq I - i$ and $k \neq K - k$ by $|I - i| \geqslant d$ and $|K - k| \geqslant d$.

2. The theorem can be generalized to more than two dimensions. For example:

$$G(L, M, N) = \begin{cases} G(M, N) & \text{if } G(L) = 0, \\ 1 & \text{if } G(L) > 0,\ G(M) > 0,\ G(N) > 0. \end{cases}$$

   The proof is completely analogous to the one given above.

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