A NOTE ON PASCAL-T TRIANGLES, MULTINOMIAL COEFFICIENTS, AND PASCAL PYRAMIDS

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1. INTRODUCTION

In what follows we give a formula for the entries in the Pascal-T triangle T_m in terms of the multinomial coefficients; this is the counterpart for these arrays of the result of Philippou [1] on the elements of the Fibonacci k-sequences in terms of the multinomial coefficients. The proof is direct, and the method also leads to a recurrence relation which gives the elements of a given triangle T_m as a combination of certain elements of the "preceeding" triangle T_{m-1}, the coefficients in the combination being binomial coefficients. Finally, because the multinomial coefficients provide the connection here, we offer some remarks on those arrays of multinomial coefficients referred to in the literature as "Pascal pyramids."

It will be convenient to recall the definition of the triangle T_m.

Definition 1.1: For any $m \geqslant 0$, T_m is the array whose rows are indexed by $n = 0, 1, 2, \ldots$, and columns by $k = 0, 1, 2, \ldots$, and whose entries are obtained as follows:

a) T_0 is the all-zero array;

b) T_1 is the array all of whose rows consist of a one followed by zeros;

c) T_m, $m \geqslant 2$, is the array whose $n = 0$ row is a one followed by zeros, whose $n = 1$ row is m ones followed by zeros, and any of whose entries in subsequent rows is the sum of the m entries just above and to the left in the preceeding row.

The entry in row n and column k is denoted by $C_m(n, k)$, although we note that

$$C_2(n, k) = \binom{n}{k},$$

since T_2 is *the* Pascal Triangle. There will be $n(m - 1) + 1$ nonzero entries in row n, and the principal property we need is that these are the coefficients (see, e.g., [2], p. 66) in the expansion

$$(1 + t + t^2 + \cdots + t^{m-1})^n = \sum_{k=0}^{n(m-1)} C_m(n, k)t^k. \tag{1.1}$$

Although it is easy to use property (c) to build the array T_m by means of the relation

$$C_m(n, k) = \sum_{j=0}^{m-1} C_m(n - 1, k - j), \tag{1.2}$$

the main result presented here evaluates $C_m(n, k)$ directly as a sum of certain multinomial coefficients.

2. FORMULAS FOR $C_m(n, k)$

Theorem 2.1: If $C_m(n, k)$ is the (n, k)-entry in T_m, $m \geqslant 3$, then for any $n \geqslant 0$ and $0 \leqslant k \leqslant n(m-1)$,

$$C_m(n, k) = \sum_{n_1, n_2, \ldots, n_m} \binom{n}{n_1, n_2, \ldots, n_m}, \tag{2.1}$$

where the summation is over all m-part compositions $n_1, n_2, \ldots, n_m$ of n such that (1) $n_1 + n_2 + \cdots + n_m = n$, and (2) $0n_1 + 1n_2 + \cdots + (m-1)n_m = k$.

Proof: The proof follows directly from the multinomial theorem, for if in

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, \ldots, n_m} x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}, \tag{2.2}$$

where the summation is over all m-part compositions of n, we put $x_i = t^{i-1}$, $1 \leqslant i \leqslant m$, we have

$$(1 + t + \cdots + t^{m-1})^n = \sum \binom{n}{n_1, \ldots, n_m} t^{n_2 + 2n_3 + \cdots + (m-1)n_m}, \tag{2.3}$$

and when the coefficients of t^k on the right-hand sides of (1.1) and (2.3) are equated, (2.1) follows from conditions (1) and (2).

Example: $C_4(4, 4) = \binom{4}{0, 4, 0, 0} + \binom{4}{1, 2, 1, 0} + \binom{4}{2, 0, 2, 0} + \binom{4}{2, 1, 0, 1}$
$= 1 + 12 + 6 + 12 = 31$

Another application of the multinomial expansion, used partly as a binomial expansion, gives the following theorem.

Theorem 2.2:

$$C_m(n, k) = \sum_{j=0}^{n} \binom{n}{j} C_{m-1}(j, k-j). \tag{2.4}$$

Proof: If the left side of (2.2) is grouped as $[x_1 + (x_2 + \cdots + x_m)]^n$, expanded as a binomial, and again t^{i-1} is substituted for x_i, the result is

$$\sum_{k=0}^{n(m-1)} C_m(n, k) t^k = \sum_{j=0}^{n} \binom{n}{j} t^j (1 + t + \cdots + t^{m-2})^j. \tag{2.5}$$

But then the factors $(1 + t + \cdots + t^{m-2})^j$ may be expressed in terms of C_{m-1}'s, using (1.1). When the coefficients of a given power of t on the right of (2.5) are collected and equated to the corresponding coefficient on the left, then (2.4) follows.

Example: $C_4(4, 4) = \binom{4}{0} C_3(0, 4) + \binom{4}{1} C_3(1, 3) + \binom{4}{2} C_3(2, 2) + \binom{4}{3} C_3(3, 1) + \binom{4}{4} C_3(4, 0)$
$= 1 \cdot 0 + 4 \cdot 0 + 6 \cdot 3 + 4 \cdot 3 + 1 \cdot 1 = 31$

3. THE PASCAL PYRAMID

The device used in the previous theorem of bracketing off one term of a multinomial in order to expand the result as a binomial can, of course, be repeated with the remaining multinomial parts, eventually running the unexpanded part down to a binomial itself; this offers the possibility of obtaining the multinomials entirely as products of binomial coefficients. In fact, this has been done in [3] and [4] for a trinomial expansion, with the multinomial coefficients appearing in the successive powers of $(x_1 + x_2 + x_3)$ being associated with points in triangular arrays, which form successive levels of a pyramidal structure—the so-called Pascal pyramid. For example, Figure 1 shows the first four levels, with each point labelled with both a multinomial coefficient and the composition which gives rise to it (the compositions can be obtained by designating the sides as first, second, third in some fashion, and letting n_1, n_2, n_3 in the composition measure units of perpendicular distances from the first, second, third sides, respectively). The law of formation for this trinomial case is clear (and also correct, as is easily verified by doing the reduction described earlier): just generate the ordinary Pascal triangle down to level n, and then multiply the rows successively upward by the numbers found in the last line. For $n = 3$, e.g.,

$$\begin{matrix} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{matrix} \qquad \text{becomes} \qquad \begin{matrix} & & & \underline{1}\cdot 1 & & & \\ & & \underline{3}\cdot 1 & & \underline{3}\cdot 1 & & \\ & \underline{3}\cdot 1 & & \underline{3}\cdot 2 & & \underline{3}\cdot 1 & \\ 1 & & 3 & & 3 & & 1 \end{matrix}$$

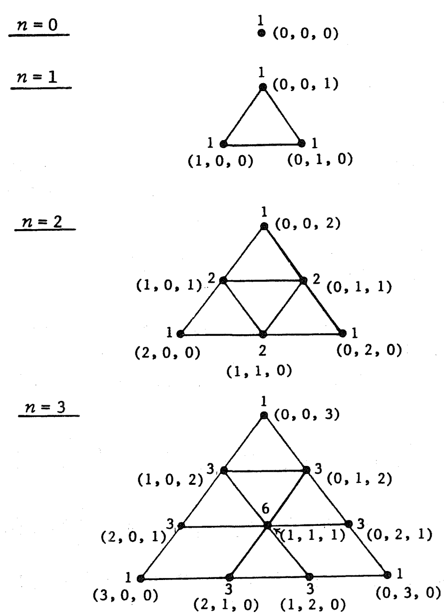

Figure 1. Levels of the Pascal Pyramid for the Case $(x_1 + x_2 + x_3)$

This works nicely, the idea is not restricted to trinomials (the generalization is not just a bigger triangle, but it is similarly simple), and there have been several comments to the effect that it is surprising that the Pascal pyramids (or hyperpyramids) are not more widely known or used.

Why should this be? An answer would seem to be that as soon as one gets past the trinomials, the method, while still elegant, becomes computationally unwieldy. That is, in the usual expansion of a multinomial $(x_1 + \cdots + x_m)^n$ there are $\binom{n + m - 1}{n}$ terms corresponding to the m-part compositions of n. To see what is required to deal with these in terms of products of binomials, we look at what might be called the Pascal square, in which we tabulate for m = 0, 1, 2, ... and n = 0, 1, 2, ... the number of m-part compositions of n (taking the entry for $m = n = 0$ to be one). The first several lines are shown below:

Pascal Square

No. Parts m	n = 0	1	2	3	4	5
0	1	0	0	0	0	0
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126

Here, the law of formation is that each entry is the sum of *all* those entries above and to the left of it in the preceeding row, and we recognize the $m = 3$ row as the triangular numbers, the $m = 4$ as the pyramidal numbers, and so on. The point here is that the square shows that the trinomials ($m = 3$) are simple sequences of products of binomials; as in the example, the ten trinomials in $(x_1 + x_2 + x_3)^3$ reduce to 4 + 3 + 2 + 1 products of binomials. But for $m > 3$, we find not sequences, but sequences of sequences. The thirty-five terms in $(x_1 + x_2 + x_3 + x_4)^4$ [the (4, 4) entry], e.g., have to be obtained using the sequence

$$\begin{aligned} 15 &= 5 + 4 + 3 + 2 + 1, \\ 10 &= 4 + 3 + 2 + 1, \\ 6 &= 3 + 2 + 1, \\ 3 &= 2 + 1, \\ 1 &= 1, \end{aligned}$$

of sequences of products of binomials. It would seem that in spite of the appeal of an array for multinomial coefficients similar to the triangle for binomials, one is better off for most purposes using a convenient algorithm (e.g., [5], pp. 46-51) to generate the m-part compositions of n, from which the exponents on the x_i and the multinomial associated with a given term are immediately available.

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Papers on all branches of mathematics and science related to the Fibonacci numbers and their generalizations are welcome. Manuscripts are requested by June 15, 1986. Abstracts and manuscripts should be send to the chairman of the local committee. Invited and contributed papers will appear in the Conference Proceedings, which are expected to be published.

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For further information concerning the conference, please contact either of the following:

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