HIGHER-ORDER FIBONACCI SEQUENCES MODULO M

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Let $\{U_n, n \geqslant 0\}$ be the ordinary Fibonacci sequence defined by

$$U_0 = 0,\ U_1 = 1,\ U_n = U_{n-1} + U_{n-2},\ \text{for } n \geqslant 2.$$

For any integer $k \geqslant 2$, let $\{V_k(n), n \geqslant -k + 2\}$ be the k^{th}-order Fibonacci sequence defined by

$$V_k(j) = 0,\ \text{for } -k + 2 \leqslant j \leqslant 0,\quad V_k(1) = 1,$$

and $$V_k(n) = V_k(n - 1) + V_k(n - 2) + \cdots + V_k(n - k),\ \text{for } n \geqslant 2.$$

It is well known that, for any integer $m \geqslant 2$, the sequence U_n $[=V_2(n)]$ mod m is periodic, and it is easy to see that this also holds for any sequence $V_k(n)$ mod m with $k \geqslant 3$. For any $m \geqslant 2$, let $p(k, m)$ denote the length of the period of the sequence $V_k(n)$ mod m. The proof of the next result is almost identical to that in [3] for the ordinary Fibonacci sequence $V_2(n)$, thus is omitted here.

Theorem 1: The sequence $V_k(n)$ mod m is simply periodic, i.e., it is periodic and it repeats by returning to its starting values. If m has the prime factorization $m = \pi q_i^{s_i}$, then $p(k, m) = \text{lcm}\,[p(k, q_i^{s_i})]$, the least common multiple of the $p(k, q_i^{s_i})$.

In order to prove Theorem 2, we first state Lemma 1, the proof of which is quite simple and, therefore, will be omitted here.

Lemma 1: Let $\{W_i(n), n \geqslant 0\}$, $i = 1, 2, 3$, be three sequences such that for each i, $W_i(n) = W_i(n - 1) + \cdots + W_i(n - k)$ for all $n \geqslant k$. If the equality $W_3(n) = W_1(n) + W_2(n)$ holds for $0 \leqslant n \leqslant k - 1$, it also holds for all $n \geqslant k$.

The following result extends the corresponding result [3] for the sequence $V_2(n)$ to any sequence $V_k(n)$ with $k \geqslant 2$. Our proof is quite different from that in [3], and we do not have a general formula for $V_k(n)$.

Theorem 2: Let q be any prime number. If $p(k, q^2) \neq p(k, q)$, then

$$p(k, q^e) = q^{e-1}p(k, q) \tag{1}$$

for any integer $e \geqslant 2$.

Proof: Let $r = p(k, q)$. For the sake of simpler notation, we shall prove (1) only for $e = 2$. The same proof stands, with obvious modifications, for $e > 2$.
Define the k-tuple

$$T_0 = (V_k(-k + 2), \ldots, V_k(1)) = (0, \ldots, 0, 1),$$

and

$$T_1 = (V_k(-k+2+r), \ldots, V_k(1+r)) = (0, \ldots, 0, 1) \mod q$$
$$= (qs_1, \ldots, qs_{k-1}, qs_k + 1) \mod q^2,$$

where $0 \leqslant s_j < q$ for $1 \leqslant j \leqslant k$, and $s_1 + \cdots + s_k \geqslant 1$. The k-tuple T_1 is obtained by moving T_0 r units to the right.

T_1 can be decomposed as follows:

$$T_1 = qs_1(1, 1, 2, \ldots, 2^{k-2}) + q(s_2 - s_1)(0, 1, 1, \ldots, 2^{k-3})$$
$$+ q(s_3 - s_2 - s_1)(0, 0, 1, \ldots, 2^{k-4}) + \cdots$$
$$+ [q(s_k - s_{k-1} - \cdots - s_1) + 1](0, 0, \ldots, 0, 1) \mod q^2.$$

Applying Lemma 1, one can obtain the k-tuple T_2 by moving T_1 r units to the right.

$$T_2 = [q(s_k - s_{k-1} - \cdots - s_1) + 1](qs_1, qs_2, qs_3, \ldots, qs_{k-1}, qs_k + 1) + \cdots$$
$$+ q(s_2 - s_1)(qs_{k-1}, qs_k + 1, q(s_k + s_{k-1} + s_{k-2}) + 1, \ldots, q(\ldots) + 2^{k-3})$$
$$+ qs_1(qs_k + 1, q(s_k + s_{k-1} + s_{k-2}) + 1, \ldots, q(\ldots) + 2^{k-2}) \mod q^2$$
$$= (2qs_1, 2qs_2, \ldots, 2qs_{k-1}, 2qs_k + 1) \mod q^2.$$

Similarly, one has

$$T_j = (jqs_1, jqs_2, \ldots, jqs_{k-1}, jqs_k + 1) \mod q^2$$

for $2 \leqslant j \leqslant q$. Since q is a prime number, $T_j \neq T_0$ for $1 \leqslant j \leqslant q - 1$, and since $T_q = T_0 \mod q^2$, we have $p(k, q^2) = qr = qp(k, q)$. This completes the proof.

As a final remark, we note that some simple facts about higher-order Fibonacci sequences can be easily observed. For example, many moduli m have the property that the sequence $V_k(n) \mod m$ contains a complete system of residue modulo m, while $m = 8$ and $m = 9$ are the smallest moduli which do not have this property in the case $k = 3$, and they are said to be defective [2]. For $m = 2$ and $m = 11$, the sequence $V_3(n) \mod m$ is uniformly distributed. (See [1] for a definition.) It is interesting to extend the results for ordinary Fibonacci sequences to those of higher order.

REFERENCES

1. L. Kuipers & J. Ś. Shiue. "A Distribution Property of the Sequence of Fibonacci Numbers." *The Fibonacci Quarterly* **10** (1972):375-376, 392.
2. A. P. Shah. "Fibonacci Sequence Modulo *m*." *The Fibonacci Quarterly* **6** (1968):139-141.
3. D. D. Wall. "Fibonacci Series Modulo *m*." *Amer. Math. Monthly* **67** (1960): 525-532.

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