ON FIBONACCI BINARY SEQUENCES

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A Fibonacci binary sequence of degree k is defined as a $\{0, 1\}$-sequence such that no $k + 1$ 1's are consecutive. For $k = 1$, we have ordinary Fibonacci sequences [2]. Let $G(k, n)$ denote the number of Fibonacci binary sequences of degree k and length n. As was given in [2], it can be easily verified that for $k = 1$, we have $G(1, 1) = 2 = F_2$, $G(1, 2) = 3 = F_3$, and $G(1, n) = G(1, n - 1) + G(1, n - 2) = F_{n+1}$ for $n \geqslant 3$, where F_n is the n^{th} Fibonacci number. In general, we have

$$G(k, n) = 2^n \qquad \text{for } 1 \leqslant n \leqslant k, \tag{1a}$$

$$G(k, n) = \sum_{j=1}^{k+1} G(k, n - j) \qquad \text{for } n \geqslant k + 1. \tag{1b}$$

Thus, for any $k \geqslant 1$, the sequence $\{F_{k,n} = G(k, n - 1),\ n \geqslant 0\}$ is the k^{th}-order Fibonacci sequence, where we set $G(k, -1) = G(k, 0) = 1$ for convenience.

Let $W(k, n)$ denote the total number of 1's in all binary sequences of degree k and length n. Then,

$$W(k, n) = n2^{n-1} \qquad \text{for } 0 \leqslant n \leqslant k, \tag{2a}$$

$$W(k, n) = \sum_{j=0}^{k} [W(k, n - j - 1) + jG(k, n - j - 1)] \text{ for } n \geqslant k + 1. \tag{2b}$$

The ratio $q(k, n) = W(k, n)/nG(k, n)$ gives the proportion of 1's in all the binary sequences of degree k and length n. It was proved in [2] that the limit

$$q(k) = \lim_{n \to \infty} q(k, n),$$

which is the asymptotic proportion of 1's in Fibonacci binary sequences of degree k, exists for $k = 1$, and actually the limit is $q(1) = (5 - \sqrt{5})/10$. It is interesting to extend this result and solve the problem for all integers $k \geqslant 1$.

Let $\{A(n),\ n \geqslant -(k + 1)\}$ be a sequence of numbers with $A(j) = 0$ for $-(k + 1) \leqslant j \leqslant -1$. If we define a sequence

$$B(n) = A(n) - A(n - 1) - \cdots - A(n - k - 1) \quad \text{for } n \geqslant 0,$$

similar to the result in the case $k = 1$, we have the inverse relation

$$A(n) = \sum_{j=0}^{k} G(k, j - 1)B(n - j) \qquad \text{for } n \geqslant 0, \tag{3}$$

where the sequence $\{G(k, n),\ n \geqslant -1\}$ is defined above. From (2), we obtain

$$\sum_{m=0}^{k} mG(k, n-m-1) = W(k, n) - \sum_{j=0}^{k} W(k, n-j-1) \qquad \text{for } n \geqslant k + 1.$$

The inverse relation (3) then implies that

$$W(k, n) = \sum_{j=0}^{n} \left(G(k, j-1) \left[\sum_{m=0}^{k} mG(k, n-j-m-1) \right] \right) \quad \text{for } n \geqslant k + 1, \tag{4}$$

where we set $G(k, n) = 0$ for $n \leqslant -2$ for convenience.

The characteristic equation for the recursion (1) is

$$h(x) = x^{k+1} - x^k - \cdots - x - 1 = 0. \tag{5}$$

Let $r_1, \ldots, r_{k+1}$ be its solution. We have the expression

$$G(k, n) = \sum_{j=1}^{k+1} c_j r_j^n, \quad \text{for } n \geqslant 0, \tag{6}$$

where c_j are constants [1]. It is known that (5) has exactly one solution, say r_1, whose norm is not less than 1 [3]. Using Cramer's rule, we obtain an explicit form for c_1 from (6):

$$c_1 = \frac{(2 - r_2)(2 - r_3)\ldots(2 - r_{k+1})(r_2 - r_3)\ \ldots\ (r_k - r_{k+1})}{(r_1 - r_2)(r_1 - r_3)\ \ldots\ (r_1 - r_{k+1})(r_2 - r_3)\ \ldots\ (r_k - r_{k+1})}$$

$$= \frac{(2 - r_2)\ \ldots\ (2 - r_{k+1})}{(r_1 - r_2)\ \ldots\ (r_1 - r_{k+1})} = \frac{1}{(2 - r_1)h'(r_1)}.$$

From the equality (4), we get

$$W(k, n) = \sum_{j=0}^{n}\left[\left(\sum_{\ell=1}^{k+1} c_\ell r_\ell^{j-1}\right)\left(\sum_{m=1}^{k} m\left(\sum_{p=1}^{k+1} c_p r_p^{n-j-m-1}\right)\right)\right].$$

Since $|r_\ell| < 1$ for $2 \leqslant \ell \leqslant n + 1$ and $G(k, n) = 0(r_1^n)$, we have

$$\sum_{j=0}^{n} c_\ell r_\ell^{j-1} m c_p r_p^{n-j-m-1}$$

$$= \begin{cases} mc_\ell c_p r_\ell^{-1} r_p^{-m-1}(r_p^{n+1} - r_\ell^{n+1})(r_p - r_\ell)^{-1} = o(nG(k, n)) & \text{for } r_p \neq r_\ell \\ nmc_\ell c_p r_\ell^{n-m-2} = o(nG(k, n)) & \text{for } r_\ell = r_p \neq r_1. \end{cases}$$

Thus,

$$q(k) = \lim_{n\to\infty}\frac{W(k, n)}{nG(k, n)} = \lim_{n\to\infty}\sum_{j=0}^{n} c_1 r_1^{j-1}\left(\sum_{m=1}^{k} mc_1 r_1^{n-j-m-1}\right)\Big/ nc_1 r_1^n$$

$$= c_1 \sum_{m=1}^{k} m r_1^{-m-2}.$$

We have established the rollowing result.

Theorem: Let r be the solution of (5) in the interval (1, 2). Then

$$q(k) = \left(\sum_{j=1}^{k} j r^{-j-2}\right)\Big/\left[(2 - r)\left((k + 1)r^k - \sum_{j=1}^{k} j r^{j-1}\right)\right].$$

Finally, three numerical examples are presented below.

For $k = 2$, $r = 1.83929$, $q(2) = 0.38158$.
For $k = 3$, $r = 1.92756$, $q(3) = 0.43366$.
For $k = 4$, $r = 1.96595$, $q(4) = 0.46207$.

REFERENCES

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