ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to Dr. A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-568 *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_1, L_1),\ (F_3, L_3),\ \ldots,\ (F_{2n+1}, L_{2n+1}),\ \ldots.$$

B-569 *Proposed by Wray G. Brady, Slippery Rock University, Slippery Rock, PA*

Find a simple curve passing through all of the points

$$(F_0, L_0),\ (F_2, L_2),\ \ldots,\ (F_{2n}, L_{2n}),\ \ldots.$$

B-570 *Proposed by Herta T. Freitag, Roanoke, VA*

Let a, b, and c be the positive square roots of F_{2n-1}, F_{2n+1}, and F_{2n+3}, respectively. For $n = 1, 2, \ldots$, show that

$$(a + b + c)(-a + b + c)(a - b + c)(a + b - c) = 4.$$

B-571 *Proposed by Heinz-Jürgen Seiffert, Student, Berlin, Germany*

Conjecture and prove a simple expression for

$$\sum_{r=0}^{[n/2]} \frac{n}{n-r}\binom{n-r}{r}$$

where $[n/2]$ is the largest integer m with $2m \leqslant n$.

B-572 *Proposed by Ambati Jaya Krishna, Student, Johns Hopkins University, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC*

Evaluate the continued fraction:

$$1 + \cfrac{2}{3 + \cfrac{4}{5 + \cfrac{6}{7 + \cdots}}}$$

B-573 *Proposed by Charles R. Wall, Trident Technical College, Charleston, SC*

For all nonnegative integers n, prove that

$$\sum_{k=0}^{n} \binom{n}{k} L_k L_{n-k} = 4 + 5 \sum_{k=0}^{n} \binom{n}{k} F_k F_{n-k}.$$

SOLUTIONS

Congruence Modulo 12

B-544 *Proposed by Herta T. Freitag, Roanoke, VA*

Show that $F_{2n+1}^2 \equiv L_{2n+1}^2 \pmod{12}$ for all integers n.

Solution by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy

First we rewrite the statement as

$$L_{2n+1}^2 - F_{2n+1}^2 \equiv 0 \pmod{12}, \tag{1}$$

then using Hoggatt's I_{18} and I_{17}, we obtain

$$L_{2n+1}^2 - F_{2n+1}^2 = 4(F_{2n+1}^2 - 1) = 4(F_{2n+1} + 1)(F_{2n+1} - 1).$$

Since $F_{2n+1} \equiv \pm 1 \pmod 3$, it is apparent that congruence (1) is satisfied for all integers n.

Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, A. F. Horadam, L. Kuipers, Bob Prielipp, M. Robert Schumann, Heinz-Jürgen Seiffert, A. G. Shannon, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Congruences Modulo 5

B-545 Proposed by Herta T. Freitag, Roanoke, VA

Show that there exist integers a, b, and c such that

$$F_{4n} \equiv an \pmod 5 \quad \text{and} \quad F_{4n+2} \equiv bn + c \pmod 5$$

for all integers n.

Solution by Hans Kappus, Rodersdorf, Switzerland

We prove by induction that for $n = 0, 1, 2, \ldots$

$$F_{4n} \equiv 3n \pmod 5, \tag{1}$$

$$F_{4n+2} \equiv 2n + 1 \pmod 5. \tag{2}$$

This is obviously true for $n = 0, 1, 2$. Assume (1) and (2) hold for some $n \geqslant 2$. Then for this n,

$$F_{4n+1} \equiv F_{4n+2} - F_{4n} \equiv 4n + 1 \pmod 5,$$

$$F_{4n+3} \equiv 2F_{4n+2} - F_{4n} \equiv n + 2 \pmod 5,$$

therefore

$$F_{4(n+1)} \equiv 3F_{4n+2} - F_{4n} \equiv 3(n + 1) \pmod 5,$$

hence (1) is true for all n. Furthermore,

$$F_{4(n+1)+2} \equiv 2F_{4(n+1)} + F_{4n+3} \equiv 2(n + 1) + 1 \pmod 5.$$

The proof is now finished.

Also solved by Wray G. Brady, Paul S. Bruckman, L. A. G. Dresel, A. F. Horadam, L. Kuipers, Bob Prielipp, Heinz-Jürgen Seiffert, A. G. Shannon, Sahib Singh, J. Suck, and the proposer.

Fibonacci Combinatorial Problem

B-546 *Proposed by Stuart Anderson, East Texas State University, Commerce, TX and John Corvin, Amoco Research, Tulsa, OK*

For positive integers a, let S_a be the finite sequence $a_1, a_2, \ldots, a_n$ defined by

$$a_1 = a,$$
$$a_{i+1} = a_i/2 \text{ if } a_i \text{ is even, } a_{i+1} = 1 + a_i \text{ if } a_i \text{ is odd,}$$

the sequence terminates with the earliest term that equals 1.

For example, S_5 is the sequence 5, 6, 3, 4, 2, 1, of six terms. Let K_n be the number of positive integers a for which S_a consists of n terms. Does K_n equal something familiar?

Solution by Piero Filipponi, Fdn. U. Bordoni, Rome, Italy

It is evident that the only sequence of length 1 is S_1, the only sequence of length 2 is S_2, and the only sequence of length 3 is S_4. That is, we have

$$k_1 = k_2 = k_3 = 1. \tag{1}$$

Let us read the sequences in reverse order so that a is the n^{th} term of S_a. By definition, a sequence S_a^n (of length n) can generate exactly one(two) sequence(s) S_a^{n+1} of length $n + 1$, if a is odd(even). Denoting by $\mathrm{e}(S_a^n)$ and $\mathrm{o}(S_a^n)$ the number of sequences of length n ending with an even term and with an odd term, respectively, we can write

$$\mathrm{e}(S_a^{n+1}) = k_n$$
$$\mathrm{o}(S_a^{n+1}) = \mathrm{e}(S_a^n) = k_{n-1},$$

from which we obtain

$$k_{n+1} = e(S_a^{n+1}) + o(S_a^{n+1}) = k_n + k_{n-1}. \tag{2}$$

From (1) and (2), it is readily seen that

$$k_n = F_{n-1}, \text{ for } n > 1.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Ben Freed & Sahib Singh, Hans Kappus, L. Kuipers, Graham Lord, J. Suck, and the proposer.

Return Engagement

B-547 *Proposed by Philip L. Mana, Albuquerque, NM*

For positive integers p and n with p prime, prove that

$$L_{np} \equiv L_n L_p \pmod{p}.$$

Solution by Sahib Singh, Clarion University of Pennsylvania, Clarion, PA

This result has been proved in B-182 (*The Fibonacci Quarterly*, 1970).

Also solved by Paul S. Bruckman, Odoardo Brugia & Piero Filipponi, L. A. G. Dresel, L. Kuipers, Bob Prielipp, Lawrence Somer, J. Suck, and the propower.

Number of Squares Needed

B-548 *Proposed by Valentina Bakinova, Rondout Valley, NY*

Let $D(n)$ be defined inductively for nonnegative integers n by $D(0) = 0$ and $D(n) = 1 + D(n - [\sqrt{n}]^2)$, where $[x]$ is the greatest integer in x. Let n_k be the smallest n with $D(n) = k$. Then

$$n_0 = 0, \quad n_1 = 1, \quad n_2 = 2, \quad n_3 = 3, \quad \text{and} \quad n_4 = 7.$$

Describe a recursive algorithm for obtaining n_k for $k \geqslant 3$.

Solution by L. A. G. Dresel, University of Reading, England

Let $[\sqrt{n}] = q$, so that $q^2 \leqslant n \leqslant (q+1)^2 - 1$, and let $R(n) = n - q^2$, so that we have $0 \leqslant R(n) \leqslant 2q$. Suppose now that n is the smallest integer for which $R(n) = r$, and consider the case where r is odd. Then we have $r = 2q - 1$ and

$$n = (q+1)^2 - 2 = \frac{1}{4}(r+3)^2 - 2.$$

By definition, we have

$$D(n_{k+1}) = k + 1$$

and

$$D(n_{k+1}) = 1 + D(R(n_{k+1})),$$

therefore

$$D(R(n_{k+1})) = k.$$

Since n_k is the smallest n for which $D(n) = k$, it follows that

n_{k+1} is the smallest n for which $R(n) = n_k$.

Now taking the case where $n \equiv 3 \pmod 4$, this leads to

$$n_{k+1} \equiv \frac{1}{4}(n_k + 3)^2 - 2$$

and we have also $n_{k+1} \equiv 3 \pmod 4$. Hence, starting with $n_3 = 3$, we can use the above recursive algorithm for $k \geqslant 3$.

Also solved by Paul S. Bruckman, Hans Kappus, L. Kuipers, Jerry M. Metzger, Sahib Singh, and the proposer.

Generalized Fibonacci Numbers

B-549 *Proposed by George N. Philippou, Nicosia, Cyprus*

Let $H_0, H_1, \ldots$ be defined by $H_0 = q - p$, $H_1 = p$, and $H_{n+2} = H_{n+1} + H_n$ for $n = 0, 1, \ldots$. Prove that, for $n \geqslant m \geqslant 0$,

$$H_{n+1}H_m - H_{m+1}H_n = (-1)^{m+1}[pH_{n-m+2} - qH_{n-m+1}].$$

Solution by L. A. G. Dresel, University of Reading, England

Define $D(n, m) = H_{n+1}H_m - H_{m+1}H_n$. Then

$$\begin{aligned} D(n, m) &= H_m(H_n + H_{n-1}) - H_n(H_m + H_{m-1}) \\ &= H_mH_{n-1} - H_nH_{m-1} = -D(n-1, m-1). \end{aligned}$$

Repeating this reduction step a further $m - 2$ times, we obtain

$$\begin{aligned} D(n, m) &= (-1)^{m-1}D(n - m + 1, 1) \\ &= (-1)^{m+1}(H_{n-m+2}H_1 - H_2H_{n-m+1}) \\ &= (-1)^{m+1}(pH_{n-m+2} - qH_{n-m+1}). \end{aligned}$$

Also solved by Paul S. Bruckman, Piero Filipponi, Herta T. Freitag, A. F. Horadam, L. Kuipers, Bob Prielipp, A. G. Shannon, P. D. Siafarikas, Sahib Singh, J. Suck, and the proposer.

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