ON THE COEFFICIENTS OF A RECURSION RELATION FOR THE FIBONACCI PARTITION FUNCTION

TAD WHITE
(Student)
California Institute of Technology, Pasadena, CA 91126
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Let $F = \{1, 2, 3, 5, \ldots\}$ be the set of Fibonacci numbers, where $F_2 = 1$, $F_3 = 2$, and thereafter $F_n = F_{n-1} + F_{n-2}$. We shall examine the function $p_F(n)$, which we define to be the number of ways to additively partition the integer n into (not necessarily distinct) Fibonacci numbers.

We first consider the generating function for $p_F(n)$. By elementary partition theory, we have

$$\sum_{n \geq 0} p_F(n)x^n = \prod_{a \in F} \frac{1}{1 - x^a} = \prod_{m \geq 2} \frac{1}{1 - x^{F_m}}. \tag{1}$$

Equivalently,

$$\left(\prod_{m \geq 2} (1 - x^{F_m})\right) \sum_{n \geq 0} p_F(n)x^n = 1. \tag{2}$$

We may expand the infinite product as a power series

$$\prod_{m \geq 2} (1 - x^{F_m}) = \sum_{m \geq 0} a_m x^m, \tag{3}$$

where a_m counts the number of partitions of m into an even number of distinct Fibonacci numbers, minus the number of partitions of m into an odd number of distinct Fibonacci numbers; we may write this as

$$a_m = p_F^e(m) - p_F^o(m). \tag{4}$$

We shall see later that knowledge of the terms a_m will lead us to a recursion relation for $p_F(n)$. With this objective in mind, we prove the following.

Theorem: Let

$$P_k = \prod_{m=2}^{k} (1 - x^{F_m}) \quad \text{when } k \geq 2,$$

and set $P_1 = 1$. Let $L_n = F_{n+1} + F_{n-1}$ be the n^{th} Lucas number. Then

$$P_\infty = \prod_{m \geq 2} (1 - x^{F_m}) = 1 - x - x^2 + \sum_{k \geq 3} x^{L_k} P_{k-2}.$$

First proof: This proof is combinatorial in nature. First consider the partial products P_k. When expanded as a power series of the form

$$P_k = \sum_{m=0}^{k} a_m^{(k)} x^m,$$

the coefficients $a_m^{(k)}$ represent the same thing as the a_m, except that the partitions are now restricted to Fibonacci numbers not exceeding F_k. It is evident that $a_m^{(k)} = a_m$ for all m such that $0 \leqslant m < F_{k+1}$, by inspection of (4). We shall use this fact later.

We may partition an integer n into distinct Fibonacci numbers in one particular way by first writing down the largest Fibonacci number not greater than n, subtracting, and iterating this process on the difference. For example,

$$27 = 21 + 5 + 1.$$

See the references listed at the end of this paper for a more detailed discussion of these points.

For simplicity of notation, we will represent a partition of a number into distinct Fibonacci numbers as a string of 1's and 0's, with the rightmost place corresponding to $F_2 = 1$, and each succeeding place corresponding to the next Fibonacci number. In the above example,

$$27 = 1 \cdot 21 + 0 \cdot 13 + 0 \cdot 8 + 1 \cdot 5 + 0 \cdot 3 + 0 \cdot 2 + 1 \cdot 1,$$

which we may write more compactly as

$$27 : \overset{(8)}{1}\ 0\ 0\ 1\ 0\ 0\ 1,$$

where the (8) signifies that the 1 below it is the coefficient of $F_8 = 21$.

The first few terms a_m for $m = 0$, 1, 2, and 3 can be obtained by direct calculation of the first few P_k; they are seen to be 1, -1, -1, and 0, respectively. Now let $n \geqslant 3$; our objective will be to characterize the terms a_m in the range $L_n \leqslant m < L_{n+1}$. As $L_3 = 4$, this will give us a_m for every nonnegative m. Since $n \geqslant 3$, we have a partition of L_{n+1} obtained in the above manner (from here on, all partitions are into distinct Fibonacci numbers, unless otherwise stated):

$$L_{n+1} : 1\ 0\ \overset{(n)}{1}\ 0\ 0\ 0 \cdots 0$$

Thus, we have the following nine possibilities for partitions of m, where $L_n \leqslant m < L_{n+1}$:

(a) $1\ 0\ 0\ \overset{(n)}{1}\ x \cdots x$	(d) $0\ 1\ \overset{(n)}{1}\ 0\ x \cdots x$	(g) $0\ 0\ \overset{(n)}{1}\ 1\ x \cdots x$
(b) $1\ 0\ 0\ 0\ x \cdots x$	(e) $0\ 1\ 0\ 1\ x \cdots x$	(h) $0\ 0\ 1\ 0\ x \cdots x$
(c) $0\ 1\ 1\ 1\ x \cdots x$	(f) $0\ 1\ 0\ 0\ x \cdots x$	(i) $0\ 0\ 0\ x\ x \cdots x$

The x's indicate "we don't care which digits go here."

In the above list, we may find a one-to-one correspondence between the partitions in (a) and the partitions in (c). Given a partition beginning with 1 0 0, we may replace these three digits with 0 1 1. Both strings will have equal value because $F_{n+2} = F_{n+1} + F_n$. However, out of each of these pairs of partitions, one is a partition of even cardinality, whereas the other is odd, since they are different only in their first three places. Hence, these partitions will cancel each other out when we compute a_m using equation (4). Similarly, there is a one-to-one correspondence between partitions of type (b) and of type (d), and they cancel out for the same reason. Partitions of the forms (f) and (g) differ only in the positions corresponding to F_{n+1}, F_n, and F_{n-1}; they cancel each other out in the same way. Partitions of the forms (h) and (i) are excluded from possibility. To see this, recall that

$$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1. \tag{5}$$

Thus, the largest number expressible in the form (h) is

$$F_n + F_n - 2 < 2F_n.$$

But $L_n = F_{n+1} + F_{n-1}$, so

$$L_n = F_n + 2F_{n-1} > F_n + F_{n-1} + F_{n-2} = 2F_n.$$

Thus, if m is expressible in the form (h), then $m < L_n$, contrary to assumption. Similarly, if m is expressible in the form (i), then $m \leqslant F_{n+1} - 2$, which is less than F_{n+1}, which in turn is less than L_n, again a contradiction.

Therefore, the only class of partitions of m which will contribute to the right-hand side of (4) are those of the form (e). But the leftmost four places in (e) form our partition of L_n; therefore, the $x \cdots x$ in (e) must represent a partition of $m - L_n$ into distinct Fibonacci numbers of size less than or equal to $n - 2$. Conversely, given any such partition of $m - L_n$, we can construct a partition of m of the form (e). Since both partitions in this correspondence are of the same parity, i.e., either both are partitions into odd numbers of Fibonacci numbers or both are partitions into even numbers of Fibonacci numbers, we deduce from (4) that

$$a_{L_n+m} = a_m^{(n-2)}, \quad \text{whenever } 0 \leqslant m < L_{n-1}. \tag{6}$$

We have thus proved the theorem.

Second proof: This proof is analytical. We require the following two results:

Let $A = a_1, a_2, \ldots$ be an arbitrary set of positive integers. If $|q| < 1$, then

$$\prod_{a \in A} \frac{1}{(1 - zq^a)} = 1 + \sum_{i \geqslant 1} \frac{zq^{a_i}}{(1 - zq^{a_1})(1 - zq^{a_2}) \cdots (1 - zq^{a_i})}, \tag{7}$$

and

$$\prod_{a \in A} (1 + zq^a) = 1 + \sum_{i \geqslant 1} (1 + zq^{a_1}) \cdots (1 + zq^{a_{i-1}}) zq^{a_i}. \tag{8}$$

Proof of (7): We consider the partial products. Clearly,

$$\frac{1}{(1 - zq^{a_1})} = 1 + \frac{zq^{a_1}}{(1 - zq^{a_1})}.$$

Now suppose that

$$\prod_{i=1}^{n} \frac{1}{(1 - zq^{a_1})} = 1 + \sum_{i=1}^{n} \frac{zq^{a_i}}{(1 - zq^{a_1})(1 - zq^{a_2}) \cdots (1 - zq^{a_i})}.$$

Then

$$\prod_{i=1}^{n+1} \frac{1}{(1 - zq^{a_i})} = \left(\prod_{i=1}^{n} \frac{1}{(1 - zq^{a_i})}\right)\left(1 + \frac{zq^{a_{n+1}}}{(1 - zq^{a_{n+1}})}\right)$$

$$= 1 + \sum_{i=1}^{n} \frac{zq^{a_i}}{(1 - zq^{a_1}) \cdots (1 - zq^{a_i})} + \frac{zq^{a_{n+1}}}{(1 - zq^{a_{n+1}})} \prod_{i=1}^{n} \frac{1}{(1 - zq^{a_i})}$$

$$= 1 + \sum_{i=1}^{n+1} \frac{zq^{a_i}}{(1 - zq^{a_1}) \cdots (1 - zq^{a_i})}.$$

By induction on n, the partial products on the left-hand side of (7) are equal to the partial sums on the right-hand side. As $|q| < 1$, the sum converges. This proves (7).

Proof of (8): The first partial product is clearly the first partial sum. So suppose that

$$\prod_{i=1}^{n}(1 + zq^{a_i}) = 1 + \sum_{i=1}^{n} (1 + zq^{a_1}) \cdots (1 + zq^{a_{i-1}})zq^{a_i}.$$

Then

$$\prod_{i=1}^{n+1}(1 + zq^{a_i}) = \left(\prod_{i=1}^{n}(1 + zq^{a_i})\right)(1 + zq^{a_{n+1}})$$

$$= 1 + \sum_{i=1}^{n} (1 + zq^{a_1}) \cdots (1 + zq^{a_{i-1}})zq^{a_i} + zq^{a_{n+1}} \prod_{i=1}^{n}(1 + zq^{a_i})$$

$$= 1 + \sum_{i=1}^{n+1} (1 + zq^{a_1}) \cdots (1 + zq^{a_{i-1}})zq^{a_i}.$$

By induction on n, this proves (8).

The following argument is due to the referee.

In (8) we set $z = -1$, $q = x$, and $A = F$:

$$\begin{aligned}
\prod_{m \geq 2}(1 - x^{F_m}) &= 1 - x - x^2 + x^3 - \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_m})x^{F_{m+1}} \\
&= 1 - x - x^2 + x^3 - \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-1}})x^{F_{m+1}} \\
&\qquad + \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-1}})x^{F_{m+2}} \\
&= 1 - x - x^2 + x^3 - \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-2}})x^{F_{m+1}} \\
&\qquad + \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-2}})x^{L_m} \\
&\qquad + \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-1}})x^{F_{m+2}} \\
&= 1 - x - x^2 + \sum_{m \geq 3} (1 - x^{F_2}) \cdots (1 - x^{F_{m-2}})x^{L_m}.
\end{aligned}$$

This proves the theorem.

Corollary: For all $m \geq 0$, a_m is either 1, -1, or 0.

Proof: We have already seen that this is true for $m \leq 3$. Now the degree of P_{k-2} is $1 + 2 + 3 + \cdots + F_{k-2}$, or $F_k - 2$, which is clearly less than $L_{k+1} - L_k$, which equals L_{k-1}. What this tells us is that the polynomials on the right-hand side of the theorem add together without overlapping. Thus, we only need to show that each P_k has coefficients $a_m^{(k)} = 0$, 1, or -1 for all m.

We do this by induction on k. Clearly, it is true for $k = 1$, as $P_1 = 1$. Suppose, then, that $a_m^{(k)} = 1, -1$, or 0 for all m and all $k < n$. By the definition of P_n, it is clear that the first F_{n+1} coefficients of P_n are identical to those of P_∞; in other words,

$$a_m = a_m^{(n)} \text{ for all } m \text{ such that } 0 \leqslant m < F_{n+1}.$$

Hence, by the theorem, the first $n + 1$ terms $a_m^{(n)}$ are the coefficients of the partial products P_k with k such that $L_k < F_{n+1}$; this includes all k less than $n - 2$ because $L_{n-3} < F_{n+1} < L_{n-2}$. By the induction hypothesis, the first F_{n+1} coefficients are either 1, -1, or 0.

Now recall that P_n is a finite product of "antipalindromic" polynomials of the form $(1 - x^{F_k})$. Thus, we have

$$a_m^{(n)} = (-1)^{n+1} a_{F_{n+2}-2-m}^{(n)}, \tag{9}$$

whenever both subscripts are positive, since the degree of P_n is $F_{n+2} - 2$. But $F_{n+1} > \frac{1}{2}(F_{n+2} - 2)$, so the first half of the coefficients in P_n are 1, -1, or 0. By (9), so are the last half. By induction, all P_k's have coefficients 1, -1, or 0. By the theorem, all terms a_m are 1, -1, or 0. This proves the corollary.

By equating like terms on both sides of (2), where we have evaluated the product P as the power series (3), we obtain, for all $n \geqslant 0$:

$$a_0 p_F(n) + a_1 p_F(n-1) + \cdots + a_{n-1} p_F(1) + a_n p_F(0) = 0,$$

where $p_F(0) = 1$ in accordance with the power series (1). This yields a recursion for $p_F(n)$ with all coefficients a_k equal to 1, -1, or 0.

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