ON FIBONACCI k-ARY TREES

DEREK K. CHANG
California State University, Los Angeles, CA 90032
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In this paper we extend some results on Fibonacci binary trees to Fibonacci k-ary trees, $k \geqslant 2$. The multinomial coefficients and higher-order Fibonacci numbers are used in our study.

For any integer $k \geqslant 2$, let $\{F_n^k,\ n \geqslant 0\}$ be a sequence of integers defined by

$$F_0^k = 0,\ F_n^k = 1, \text{ for } 1 \leqslant n \leqslant k,$$

and

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k}^k, \text{ for } n \geqslant k+1.$$

The sequence $\{F_n^2,\ n \geqslant 0\}$ is thus the ordinary Fibonacci sequence. For $k \geqslant 3$, the sequence $\{F_n^k,\ n \geqslant 0\}$ is different from the Fibonacci sequence $\{V_n^k,\ n \geqslant 0\}$ of order k, which is defined by

$$V_0^k = 0,\ V_1^k = 1,\ V_n^k = 2^{n-2}, \text{ for } 2 \leqslant n \leqslant k,$$

and

$$V_n^k = V_{n-1}^k + V_{n-2}^k + \cdots + V_{n-k}^k, \text{ for } n \geqslant k+1.$$

We also need the following integer sequence. For any integer $k \geqslant 2$ and $1 \leqslant m \leqslant k$, let $\{F_n^{k,m},\ n \geqslant -k\}$ be a sequence defined by

$$F_n^{k,m} = 0, \text{ for } n \leqslant 0,$$

$$F_n^{k,m} = 2^{n-1}, \text{ for } 1 \leqslant n \leqslant m,$$

and

$$F_n^{k,m} = F_{n-1}^{k,m} + F_{n-2}^{k,m} + \cdots + F_{n-k}^{k,m}, \text{ for } n \geqslant m+1.$$

It is easy to see that, for any integer $k \geqslant 1$, the sequence $\{F_n^{k,1},\ n \geqslant 0\}$ is precisely the Fibonacci sequence of order k, i.e., $F_n^{k,1} = V_n^k$. By induction, it can be shown that, for any $k < n$,

$$\sum_{m=1}^{k} F_{n-k}^{k,m} = F_n^k.$$

For any fixed $k \geqslant 2$ and $n \geqslant 0$, one can obtain multinomial coefficients $c_{n,j}^k$, $0 \leqslant j \leqslant (k-1)n + 1$, by expanding the expression

$$(1 + x + x^2 + \cdots + x^{k-1})^n,$$

and obtain the corresponding (generalized) Pascal triangle (see [4], [5], and [6]). For convenience, we set $c_{n,j}^k = 0$, for $j \leqslant -1$ and $j \geqslant (k-1)n + 2$. For $k = 2$, one has binary coefficients and the Pascal triangle. For $k = 3$, one has trinomial coefficients $c_{n,j}^3$ and the corresponding generalized Pascal triangle, as shown in Figure 1.

One can draw diagonals in the triangle, and see that the sums of numbers between parallel lines are precisely the 3rd-order Fibonacci numbers V_n^3, just as in the case $k = 2$ [1, p. 245]. In general, by an argument similar to that in [1, p. 246], one has the following relation between Fibonacci numbers V_n^k and multinomial coefficients $c_{n,j}^k$:

$$V_{n+1}^k = \sum_{j=0}^{\lfloor n - n/k \rfloor} c_{n-j,j}^k,$$

where $\lfloor\ \rfloor$ indicates the largest integer function.

1	1										
1	1	1	1								
2	1	2	3	2	1						
4	1	3	6	7	6	3	1				
7	1	4	10	16	19	16	10	4	1		
13	1	5	15	30	45	51	45	30	15	5	1

Figure 1. Generalized Pascal Triangle

For any fixed integer $k \geqslant 2$, we now define the Fibonacci trees T_n^k of order n, $n \geqslant 1$, inductively on n. For $1 \leqslant n \leqslant k$, T_n^k consists of only a root node. For $n \geqslant k + 1$, T_n^k consists of a root node with k ordered sons $T_{n-1}^k, T_{n-2}^k, \ldots, T_{n-k}^k$ from left to right. For $k = 2$, one has the ordinary Fibonacci trees [2]. For $k = 3$, one has the Fibonacci ternary trees.

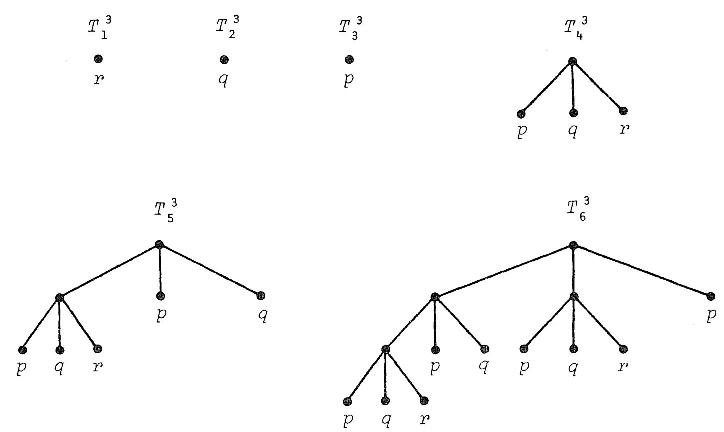

Figure 2. Fibonacci Ternary Trees

In Figure 2, every terminal node of T_n^3 is labelled by p, q, or r. For each $n \geqslant 5$, the tree T_n^3 can be obtained from T_{n-1}^3 by replacing all the labels r and q in T_{n-1}^3 by q and p, respectively, and replacing all the terminal nodes in T_{n-1}^3 with label p by T_4^3. This is a simple rule to grow a Fibonacci tree to a higher order, similar to that given in [3] for Fibonacci binary trees. One can also set a similar rule to grow the trees T_n^k with any fixed $k \geqslant 3$, where, instead of using only three labels, k labels—$p_1, p_2, \ldots, p_k$—are needed.

For any $k \geqslant 2$, a k-ary tree is a tree with each internal node containing exactly k ordered sons. We now specify branch costs of a k-ary tree. We will assume that each left-most branch has unit cost 1, each second-to-the-left branch has cost 2, ..., and each right-most branch has cost k. The cost a_i of a node i is the sum of costs of the branches from the root to this node. If the path from the root to a node has ℓ branches, the node is said to be at level ℓ. The average cost of a tree T is defined by

$$s = \sum_{j=1}^{m} a_j/m,$$

where m is the number of terminal nodes in T, and the summation is over all the terminal nodes in T. As in the case $k = 2$ (see [2]), one can see that if a k-ary tree has n internal nodes, then it has $(k - 1)n + 1$ terminal nodes. It is easy to verify the following lemma.

Lemma 1: In a k-ary tree, let a_i be the cost of the terminal node i, and let b_j be the cost of the internal node j. Then

$$s = \frac{\sum_{i=1}^{(k-1)n+1} a_i}{(k - 1)n + 1} = \frac{(k - 1)\sum_{j=1}^{n} b_j + nk(k + 1)/2}{(k - 1)n + 1}.$$

As was stated in [7], one can construct an optimal k-ary tree in the sense of minimum average cost as follows: Suppose that an optimal k-ary tree with $(k - 1)(n - 1) + 1$ terminal nodes is given. To obtain an optimal k-ary tree with $(k - 1)n + 1$ terminal nodes, one can split a terminal node of minimum cost in T to produce k new terminal nodes. This can be verified by using Lemma 1, just as was done in the case $k = 2$ in [2].

It is obvious that each tree T_n^k is a k-ary tree, and that it has F_n^k terminal nodes. As in the case $k = 2$ in [2], we have the following lemma.

Lemma 2: Each Fibonacci tree T_n^k, $n \geqslant k + 1$, has exactly $F_{n-k}^{k,j}$ terminal nodes of cost $n - j$, where $1 \leqslant j \leqslant k \leqslant n - 1$.

Proof: The proof is by induction on n. The tree T_{k+1}^k has k terminal nodes, and it has exactly 1 $(= F_1^{k,j})$ terminal node of cost $k + 1 - j$, where $1 \leqslant j \leqslant k$. Now, we assume that the Lemma holds for all n, $k + 1 \leqslant n \leqslant N$, where $N \geqslant k + 1$ is a fixed integer. The tree T_{N+1}^k has k subtrees T_N^k, T_{N-1}^k, ..., T_{N-k+1}^k, from left to right. The number of terminal nodes of cost $N + 1 - j$ in T_{N+1}^k is, for $k \geqslant j \geqslant N + 1 - k$,

$$\begin{aligned} F_{N-k}^{k,j} + F_{N-k-1}^{k,j} + \cdots + F_1^{k,j} + 1 &= 2^{N-k-1} + 2^{N-k-2} + \cdots + 1 + 1 \\ &= 2^{N-k} = F_{N+1-k}^{k,j}, \end{aligned}$$

and for $j < N + 1 - k$, the number is

$$F_{N-k}^{k,j} + F_{N-k-1}^{k,j} + \cdots + F_{N-2k+1}^{k,j} = F_{N+1-k}^{k,j}.$$

This completes the proof.

With the branch costs specified as above, for any fixed $k \geqslant 2$, we have the following theorem.

Theorem 1: The average cost of a Fibonacci tree T_n^k of order n is

$$s_n^k = (n - 1) + \left[\sum_{j=2}^{k} (j - 1)F_{n-k}^{k,j}\right] \Big/ F_n^k,$$

and it is optimal among k-ary trees for each $n \geqslant k + 1$.

Proof: By Lemma 2, for $n \geqslant k + 1$,

$$\begin{aligned} s_n^k &= \sum_{j=1}^{k} (n - j)F_{n-k}^{k,j} \Big/ F_n^k \\ &= (n - 1)\sum_{j=1}^{k} F_{n-k}^{k,j} \Big/ F_n^k - \sum_{j=1}^{k} (j - 1)F_{n-k}^{k,j} \Big/ F_n^k \\ &= (n - 1) - \sum_{j=2}^{k} (j - 1)F_{n-k}^{k,j} \Big/ F_n^k. \end{aligned}$$

If $k = 2$, one has

$$s_n^2 = (n - 1) - F_{n-2}^{2,2}/F_n^2 = (n - 2) + (F_n^2 - F_{n-1}^2)/F_n^2 = (n - 2) + F_{n-2}^2/F_n^2,$$

as was shown in Theorem 3 of [2].

For the second assertion, by the rule for constructing optimal k-ary trees mentioned above, we need to show that, by splitting all the terminal nodes of cost $(n - k)$ in T_n^k, we can obtain T_{n+1}^k. As in the proof in [2], we proceed by induction on n. The claim clearly holds for $n = k + 1$. We assume it holds for all n, $k + 1 \leqslant n \leqslant N - 1$, where $N \geqslant k + 2$ is a fixed integer. Since the leftmost subtree of T_N^k is T_{N-1}^k, by the induction hypothesis, after splitting all the terminal nodes of cost $(N - k)$ in this subtree, we obtain T_N^k. A similar argument applies to all the remaining $(k - 1)$ subtrees of T_N^k. Therefore, the resulting tree has k ordered subtrees $T_N^k, T_{N-1}^k, \ldots, T_{N-k+1}^k$, and so it is T_{N+1}^k. This completes the proof.

Our next result generalizes a result in [3] which deals with the number of terminal nodes at each level of a Fibonacci tree.

Theorem 2: At level ℓ in a Fibonacci tree T_n^k, $n \geqslant k + 1$, there are $c_{\ell,n-k-\ell}^k$ nodes with label p_1, and $c_{\ell-1,n-k-\ell}^k + c_{\ell-1,n-k-\ell-1}^k + \cdots + c_{\ell-1,n-k-\ell-(k-j)}^k$ nodes with label p_j, $2 \leqslant j \leqslant k$.

Proof: The assertion holds for $n = k + 1$. We assume that it holds for some $n \geqslant k + 1$, and then prove it for $n + 1$. By hypothesis, there are

$$c_{\ell-1,n-k-\ell+1}^k$$

nodes with label p_1 in T_n^k at level $\ell - 1$, and

$$c_{\ell-1,n-k-\ell}^k + \cdots + c_{\ell-1,n-k-\ell-(k-2)}^k$$

nodes with label p_2 in T_n^k at level ℓ. Thus, the number of nodes with label p_1 in T_{n+1}^k at level ℓ is

$$c_{\ell-1,n-k-\ell+1}^k + c_{\ell-1,n-k-\ell}^k + \cdots + c_{\ell-1,n-k-\ell-(k-2)}^k = c_{\ell,n+1-k-\ell}^k.$$

Similarly, one can compute the number of nodes with label p_j, $j \geqslant 2$, in T_{n+1}^k at level ℓ. This completes the proof.

One can see from Theorem 2 that the number of terminal nodes at level ℓ in the tree T_n^k is

$$\sum_{j=1}^{k} jc_{\ell-1,\, n-2k-\ell+j}^{k},$$

and since T_n^k has F_n^k terminal nodes,

$$F_n^k = \sum_{\ell=1}^{n-k} \sum_{j=1}^{k} jc_{\ell-1,\, n-2k-\ell+j}^{k} \text{ for } n \geqslant k+1.$$

Finally, one can see that the trees T_n^k and T_n^2 have average costs s_n^k and s_n^2, respectively. Since the characteristic equations of the recurrence relations for the sequences $\{F_n^k,\ n \geqslant 0\}$ and $\{F_n^{k,j},\ n \geqslant 0\}$ are the same, and have exactly one root x_1 satisfying $|x_1| > 1$, and since the coefficients of the n^{th} power of x_1 in the expressions of F_n^k and $F_n^{k,j}$ are clearly nonzero, the ratio $F_{n-k}^{k,j}/F_n^k$ converges to a finite limit as $n \to \infty$. Using Theorem 1, one has the limit

$$s_n^k/s_n^2 \to 1 \text{ as } n \to \infty.$$

On the other hand, the trees T_n^k and T_n^2 have F_n^k and F_n^2 terminal nodes, respectively. For any $k \geqslant 3$, one has the limit

$$F_n^k/F_n^2 \to \infty \text{ as } n \to \infty.$$

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