FIBONACCI NUMBERS AS EXPECTED VALUES IN A GAME OF CHANCE

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Our objective in this note is to introduce an interesting game of chance and show that, when the game is unfair, its expected value is (plus or minus) a Fibonacci number. We prove this in an elegant and unexpected way, with ramifications going beyond the Fibonacci numbers.

1. THE GAME

We assign five payoffs to the vertices of a pentagon. Three of these are \$0, the remaining two are $\$2^N$ and $\$-2^N$, where N is a fixed positive integer (preferably large). A ball moves clockwise around the five positions, and where it stops determines the payoff. The ball is propelled by coin tossing. When a fair coin shows a head, the ball moves one position clockwise. When the coin shows a tail, the ball does not move. The coin is tossed N times. The distribution of the payoffs, the starting position of the ball, and the value of N are immaterial to the mathematics—the Fibonacci numbers are here no matter what. As for the gambler's fortune, that is another story.

The expected value of the game is easily shown to have the form

$$\sum_j \left(\binom{N}{5j+p} - \binom{N}{5j+q}\right), \quad 0 \leqslant p,\ q \leqslant 4, \tag{1}$$

but these integers are not immediately recognizable as positive or negative, let alone Fibonacci numbers.

2. GENERALIZED BINOMIAL COEFFICIENTS

The following is a well-known identity (see, e.g., [1], Chap. 3, Prob. 29).

$$\binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \cdots + \binom{n}{n}F_n = F_{2n}, \tag{2}$$

where $\{F_j\}_{j \geqslant 0} = \{0, 1, 1, 2, \ldots\}$ is the Fibonacci sequence. There are several ways to prove (2), but here is a way which gets to the heart of the relation between the Fibonacci numbers and the binomial coefficients. Let

$$\begin{Bmatrix} n \\ j \end{Bmatrix}^* = F_{2n+j}.$$

Observe that

$$\begin{Bmatrix} n+1 \\ j \end{Bmatrix}^* = F_{2n+2+j} = F_{2n+j+1} + F_{2n+j} = \begin{Bmatrix} n \\ j+1 \end{Bmatrix}^* + \begin{Bmatrix} n \\ j \end{Bmatrix}^*. \tag{3}$$

Except for the advanced, as opposed to retarded j-argument, (3) states the Pascal recurrence for the coefficients $\begin{Bmatrix} n \\ j \end{Bmatrix}^*$. Because of the close connection to the binomial coefficients, there must be a precise statement relating the two. Leaving the particular choice of $\begin{Bmatrix} n \\ j \end{Bmatrix}^*$ values behind, but retaining recurrence

(5), below, this is

$$\sum_j \binom{n}{j}\left\{{r \atop j+i}\right\} = \left\{{n+r \atop i}\right\}, \tag{4}$$

a formula [2] easily proved by induction on n (fixed $r \geqslant 0$ and $-\infty < i < +\infty$). Setting $\left\{{k \atop j}\right\} = \left\{{k \atop j}\right\}^*$ and $r = i = 0$ in (4) yields (2).

The lesson to be learned from this is two-fold. First, (4) depends only on the Pascal-like recurrence

$$\left\{{n+1 \atop j}\right\} = \left\{{n \atop j}\right\} + \left\{{n \atop j+1}\right\} \tag{5}$$

so (2) holds for *any* sequence satisfying the Fibonacci recurrence (e.g., the Lucas numbers). We are motivated to look for more *generalized binomial coefficients* (gbc's) among the Fibonacci numbers, and find them easily:

$$\left\{{n \atop j}\right\} \text{ could be } F_{m+n-j},\ (-1)^{n+j}2^jF_{m-3n+j},\ 2^{-n-j}(-1)^jF_{m-2n+j},$$
$$(-1)^{n+j}F_{m-n+j},\ 2^{-n-j}F_{m+2n-j},\ 2^jF_{m+3n+j},\ (-1)^jF_{m-2n-j},\ \ldots .$$

Secondly, since the initial conditions

$$\left\{{0 \atop j}\right\} = c_j$$

are free for us to choose, rewriting (4) as

$$\left\{{n \atop 0}\right\} = \sum_j \binom{n}{j} c_j$$

gives us a single coefficient which computes entire binomial sums.

Thus, the idea of a generalized binomial coefficient is itself worth generalizing. Let

$$\begin{bmatrix} n \\ j \end{bmatrix} \equiv (-1)^n\left\{{n \atop j}\right\} \quad\text{and}\quad \left\langle {n \atop j} \right\rangle \equiv \begin{bmatrix} n \\ j \end{bmatrix} - \inf_k \begin{bmatrix} n \\ k \end{bmatrix},\ n \in \mathbb{N},\ j,\ k \in \mathbb{Z}. \tag{6}$$

To assure that the gbc's $\left\langle {n \atop j} \right\rangle$ are well defined, we need only require

$$\sup_k \left| \left\{{0 \atop k}\right\} \right| < +\infty.$$

3. THE GAME AND THE gbc's

By answering some natural questions about how the coefficients $\left\{{n \atop j}\right\}$, $\begin{bmatrix} n \\ j \end{bmatrix}$, and $\left\langle {n \atop j} \right\rangle$ are related, we get immediate answers about the connection between the roulette-like game of Section 1 and the Fibonacci numbers. For example, what type of recurrence do the $\left\langle {n \atop j} \right\rangle$ satisfy? Given

$$\left\{ \left\langle {m \atop k} \right\rangle \right\}_{\substack{0 \leqslant m \leqslant n \\ -\infty < k < +\infty}},$$

how do we recover $\left\{{n \atop j}\right\}$? The answers are in

Theorem 1: Let $\binom{n}{j}$ denote the binomial coefficients, and $\left\{{n \atop j}\right\}$ denote any coefficients satisfying (5), $n = 0, 1, \ldots;\ -\infty < j < +\infty$. With the convention

$$\left\{ {0 \atop j} \right\} = \left[{0 \atop j} \right] = \left\langle {0 \atop j} \right\rangle,$$

define $\left[{n \atop j} \right]$ and $\left\langle {n \atop j} \right\rangle$ by (6). Then

$$\left\langle {n+1 \atop j} \right\rangle = \lambda_{n+1} - \left\langle {n \atop j} \right\rangle - \left\langle {n \atop j+1} \right\rangle,$$
$$\text{with } \lambda_{n+1} = \sup_k\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right); \tag{7}$$

$$\left\{ {n \atop j} \right\} = (-1)^n\left\langle {n \atop j} \right\rangle + \sum_{k=1}^{n} (-1)^{k-1}2^{n-k}\lambda_k; \tag{8}$$

$$\sum_{j=0}^{n} \binom{n}{j}\left\langle {r \atop j+i} \right\rangle = (-1)^n\left\langle {n+r \atop i} \right\rangle + \sum_{k=1}^{n} (-1)^{k-1}2^{n-k}\lambda_{r+k}. \tag{9}$$

Outline of Proof: A straightforward application of the definitions yields (7).

To obtain (8), let $S_0 = 0$ and

$$S_n = \begin{cases} \inf_k\left\{ {n \atop k} \right\}, & n \text{ even} \\ \sup_k\left\{ {n \atop k} \right\}, & n \text{ odd}, n > 0. \end{cases}$$

It follows that $S_n = \left\{ {n \atop j} \right\} + (-1)^{n+1}\left\langle {n \atop j} \right\rangle$ for all j, and

$$S_{n+1} = 2S_n + (-1)^n\lambda_{n+1}. \tag{10}$$

Solving (10) gives

$$S_n = \sum_{k=1}^{n} (-1)^{k-1}2^{n-k}\lambda_k$$

and (8).

To obtain (9), substitute (8) with the appropriate indices in (4). ∎

Here are the important consequences of Theorem 1: Relation (7) is an *algorithm* for constructing an array of gbc's $\left\langle {n \atop j} \right\rangle$. Consideration of (1) shows that we will want to take

$$\ldots 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \ldots \tag{11}$$

as our initial row. Secondly, setting $r = 0$ in (9) and choosing i appropriately,

$$\Sigma_j \left(\binom{N}{5j+p} - \binom{N}{5j+q}\right) = (-1)^N\left(\left\langle {N \atop -p} \right\rangle - \left\langle {N \atop -q} \right\rangle\right). \tag{12}$$

The implication is that to know the expected value of our game we need only construct the array of gbc's $\left\langle {n \atop j} \right\rangle$ with initial row (11). Here is where the Fibonacci numbers appear.

$$\begin{array}{cc|ccccc|ccc}
\ldots & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
\ldots & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & \ldots \\
\ldots & 2 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & \ldots \\
\ldots & 0 & 2 & 3 & 2 & 0 & 0 & 2 & 3 & \ldots \\
\ldots & 3 & 0 & 0 & 3 & 5 & 3 & 0 & 0 & \ldots \\
\ldots & 5 & 8 & 5 & 0 & 0 & 5 & 8 & 5 & \ldots \\
 & \vdots & & & & & & \vdots & &
\end{array} \tag{13}$$

A complete description of array (13) is the concern of

Theorem 2: Array (13) consists of rows of repeating blocks

$$B_n = \left(\left\langle {n \atop j} \right\rangle\right)_{j=0}^{4}$$

which (modulo a shift) have the form

$$M_n = (F_{n+1}, F_n, 0, 0, F_n). \tag{14}$$

Let $R_k(\cdot)$ denote the operator which shifts the elements of a vector k steps to the right with wraparound. Then,

$$B_n = R_{2n(\text{mod } 5)} M_n, \; n = 0, 1, \ldots . \tag{15}$$

Outline of Proof: The fact that the blocks have the form $(b_n, a_n, 0, 0, a_n)$, where eventually $0 < a_n < b_n$, is a simple observation, as is the right-shifting action described by (15).

The fact that the a_n and b_n are the Fibonacci numbers follows from the basic recurrence (7). The latter implies

$$\begin{aligned} b_{n+1} &= b_n + a_n \\ a_{n+1} &= b_{n+1} - a_n, \end{aligned} \tag{16}$$

and (16) implies, in turn, that $b_{n+2} = b_{n+1} + b_n$, $a_{n+2} = a_{n+1} + a_n$. With the initial conditions, we have $b_n = F_{n+1}$, $a_n = F_n$. ∎

Corollary: The expected value of the game of Section 1 is zero or (plus or minus) a Fibonacci number.

Proof: Consider (12) in conjunction with (13), (14), (15). The difference of any two elements in (14) is zero or (plus or minus) a Fibonacci number. ∎

4. EXTENSIONS

A natural generalization of the game is to assign payoffs to the vertices of an n-gon and ask about the analogues of the Fibonacci numbers in this case. This question is addressed in [3], where we generalize results of Hoggatt and Alexanderson [4].

REFERENCES

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