THE FIBONACCI RATIO IN A THERMODYNAMICAL PROBLEM: A COMBINATORIAL APPROACH

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(Submitted March 1984)

In a previous contribution to this journal [1], the author showed how the Fibonacci ratio arises in the solution of a particular thermodynamical problem, namely, the calculation of the entropy of a chain of electrons localized onto lattice sites with density one, and with the constraints that half the lattice sites may contain at most two electrons each, while the other half may contain at most only one electron each. The use of the thermodynamical grand-canonical formulation [1], while simplifying the calculation, greatly obscured the purely combinatorial nature of the problem, which we think is by itself a fascinating one, and which we purport here to present.

The problem in [1] might be restated as follows: Given $2N$ different boxes, and $2N$ identical coins, with half the boxes containing at most two coins each, and the other half containing at most one coin each, in how many different ways can one arrange or put the $2N$ coins into the $2N$ boxes, as $N \to \infty$? Although a single coin may be put into a box in two different ways, as head or tail, we shall agree that once we put two coins into a box we shall not inquire as to which is head or which is tail, and shall count that arrangement as only one.

With this understanding, it is straightforward to show, in a purely combinatorial manner, that the total number of arrangements $A(N)$ of the $2N$ coins in the $2N$ boxes, is given by

$$A(N) = 2^{2N} \sum_{k=N}^{2N} 2^{-k} \binom{2N}{k} \binom{N}{2N-k}, \tag{1}$$

for $N = 1, 2, 3, \ldots$. The above expression is exact and holds for any $N \geqslant 1$; a proof of (1) is given in the text.

The Fibonacci ratio arises from (1) through the "entropy" $S(N)$ associated with the number of arrangements $A(N)$, i.e.,

$$S(N) \equiv \ln A(N), \tag{2}$$

and the extensive property of the entropy in the thermodynamic limit ($N \to \infty$), i.e.,

$$\lim_{N \to \infty} \frac{S(N)}{N} = \ln k, \text{ with } k \equiv f^5, \tag{3}$$

a constant independent of N, where here $f \equiv (1 + \sqrt{5})/2$ is the positive Fibonacci ratio.

In the remainder of this paper we give the proofs of Equations (1) and (3). Equation (1) can be proved by the use of the well-known generating function method [2] of combinatorial analysis.

Thus, $A(N)$ will be the coefficient of an x^{2N} in the expansion in powers of x of the appropriate generating or enumerating function,

$$G(x) = (1 + 2x)^N (1 + 2x + x^2)^N = (1 + 2x)^N (1 + x)^{2N}. \tag{4}$$

In the enumerating function $G(x)$ in (4), the enumerating factor $(1 + 2x)^N$ takes account of the N boxes that contain at most one coin each, while the enumerating factor $(1 + x)^{2N}$ takes account of the N boxes that contain at most two coins each. But,

$$G(x) = \left(\sum_{\ell=0}^{N}\binom{N}{\ell}(2x)^{\ell}\right)\cdot\left(\sum_{m=0}^{2N}\binom{2N}{m}x^{m}\right) = \sum_{\ell=0}^{N}\sum_{m=0}^{2N}2^{\ell}\binom{N}{\ell}\binom{2N}{m}x^{\ell+m}. \tag{5}$$

The coefficient of x^{2N} will thus be given by the sum of all the terms in (5), such that $\ell + m = 2N$, i.e., by

$$\sum_{m=N}^{2N}2^{2N-m}\binom{N}{2N-m}\binom{2N}{m},$$

which proves Equation (1).

We now proceed to give the proof of Equation (3). When $N \to \infty$, each term in the right-hand side of Equation (1), with $N \leqslant k \leqslant 2N$, is a product of a very rapidly increasing function of k, namely $[(2N-k)!]^{-2}$, times a very rapidly decreasing function of k, namely $[2^k k!(k-N)!]^{-1}$. In the thermodynamic limit, $N \to \infty$, the product of the two functions will have an extremely sharp maximum for some value of k, with $N \leqslant k \leqslant 2N$. In the limit $N \to \infty$, the entire right-hand side summation in Equation (1) can thus be replaced by the maximum term in the same summation, in an asymptotically exact manner.

We will then have

$$\frac{S(N)}{N} = 2\ln 2 + \frac{1}{N}\ln\left(\sum_{k=N}^{2N}P(k;N)\right), \tag{6}$$

where

$$P(k;N) \equiv 2^{-k}\binom{2N}{k}\binom{N}{2N-k},$$

and hence

$$\lim_{N\to\infty}\frac{S(N)}{N} = 2\ln 2 + \lim_{N\to\infty}\frac{1}{N}[\ln P(k;N)]_{\text{Max.}}, \tag{7}$$

where we have used $\ln[P(k;N)]_{\text{Max.}} = [\ln P(k;N)]_{\text{Max.}}$. But

$$\begin{aligned}\ln P(k;N) = {}&-k\ln 2 + \ln(2N)! - \ln k! - \ln(2N-k)!\\ &+ \ln N! - \ln(2N-k)! - \ln(k-N)!,\end{aligned} \tag{8}$$

and by the Stirling approximation, we have that (for $N \to \infty$)

$$\ln N! = N(\ln N - 1) + \frac{1}{2}\ln N + C_N, \tag{9}$$

where C_N is a number of the order of unity.

Then, we will obviously have that

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}[\ln P(k;N)]_{\text{Max.}} &= \left[\lim_{N\to\infty}\frac{1}{N}\ln P(k;N)\right]_{\text{Max.}}\\ &= \Big\{\lim_{N\to\infty}\frac{1}{N}[-k\ln 2 - k(\ln k - 1) - 2(2N-k)(\ln(2N-k) - 1)\\ &\qquad - (k-N)(\ln(k-N) - 1) + Q(N)]\Big\}_{\text{Max.}} \equiv [\mathscr{P}(k;N)]_{\text{Max.}}\,,\end{aligned} \tag{10}$$

where

$$Q(N) = 2N \ln 2 + 3N(\ln N - 1) \tag{11}$$

is a function of N only.

It is important to notice that the terms in

$$\frac{1}{2} \ln N + C_N$$

in the Stirling approximation in (9) and the corresponding ones for $(2N - k)!$, $k!$, and $(k - N)!$ contribute *nothing* to the limit in (10), since, typically,

$$\lim_{N \to \infty} \frac{1}{N}\left(\frac{1}{2} \ln N + C_N\right) = 0.$$

To find the maximum of the function $\mathscr{P}(k;\, N)$ defined in (10), we find the value of k, such that

$$\frac{d}{dk}\mathscr{P}(k;\, N) = 0, \tag{12}$$

where N is considered as a parameter. This value of k is then substituted back into $\mathscr{P}(k;\, N)$ to give $(\mathscr{P}(k;\, N))_{\text{Max.}}$.

Interchanging the derivative with the limit in (10), Equation (12) leads to

$$\begin{aligned}-\ln 2 - \ln k + 1 - \frac{k}{k} + 2(\ln(2N - k) - 1)\\ + 2\,\frac{(2N - k)}{(2N - k)} - \ln(k - N) + 1 - \frac{(k - N)}{(k - N)} = 0,\end{aligned} \tag{13}$$

or

$$-\ln 2 - \ln k + 2 \ln(2N - k) - \ln(k - N) = 0, \tag{14}$$

for k.

Finally, Equation (14) leads to

$$\frac{(2N - k)^2}{2k(k - N)} = 1,$$

or the quadratic equation

$$\left(\frac{k}{2N}\right)^2 + \left(\frac{k}{2N}\right) - 1 = 0, \tag{15}$$

for k. Because k must be a positive number, the only appropriate solution of (15) is

$$\frac{k}{2N} = \frac{1}{f}, \qquad (N \leqslant k \leqslant 2N),$$

where f is the positive Fibonacci ratio. Hence, we will have

$$\begin{aligned}(\mathscr{P}(k;\, N))_{\text{Max.}} = \lim_{N \to \infty} \frac{1}{N}\Bigg[-\frac{2N}{f} \ln 2 - \frac{2N}{f}\left(\ln\left(\frac{2N}{f}\right) - 1\right) - 2\left(2N - \frac{2N}{f}\right)\left(\ln\left(2N - \frac{2N}{f}\right) - 1\right)\\ - \left(\frac{2N}{f} - N\right)\left(\ln\left(\frac{2N}{f} - N\right) - 1\right) + Q(N)\Bigg] =\end{aligned}$$

(continued)

$$= -\frac{2}{f}\ln 2 - \frac{2}{f}\ln 2 + \frac{2}{f}\ln f + \frac{2}{f} - \frac{4}{f}(f - 1)(\ln 2 + \ln(f - 1) - \ln f - 1)$$
$$- \frac{(2 - f)}{f}(\ln(2 - f) - \ln f - 1) + 2\ln 2 - 3$$
$$= -2\ln 2 + 3\ln f + \frac{4}{f}\ln(f - 1) - 4\ln(f - 1) - \frac{2}{f}\ln(2 - f) + \ln(2 - f). \tag{16}$$

By using the relationships $\ln(f - 1) = -\ln f$ and $\ln(2 - f) = -2\ln f$, which hold true for the Fibonacci ratio f, (16) finally leads to the remarkably simple result

$$(\mathscr{P}(k; N))_{\text{Max.}} = -2\ln 2 + 5\ln f,$$

or, finally,

$$\lim_{N\to\infty}\frac{S(N)}{N} = 5\ln f.$$

This proves Equation (3) and coincides, of course, with the result obtained in [1] through the use of the grand-canonical formalism.

REFERENCES

1. J.-P. Gallinar. "Fibonacci Ratio in a Thermodynamical Case." *The Fibonacci Quarterly* **17**, no. 3 (1979):239.
2. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: Wiley, 1967, Chapter 2.

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