DETERMINANTAL HYPERSURFACES FOR LUCAS SEQUENCES OF ORDER r, AND A GENERALIZATION

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1. INTRODUCTION

In Hoggatt and Bicknell [1], the Fibonacci sequence $\{R_n\}$ of order $r\,(\geqslant 2)$ was defined by

$$R_{n+r} = R_{n+r-1} + R_{n+r-2} + \cdots + R_n, \quad R_1 = 1,\ R_2 = 1, \tag{1.1}$$

with

$$R_{-(r-2)} = R_{-(r-3)} = \cdots = -R_1 = R_0 = 0. \tag{1.2}$$

Using the method of a generating matrix for $\{R_n\}$, they obtained the determinantal identity

$$\begin{vmatrix} R_{n+r-1} & R_{n+r-2} & \cdots & R_{n+1} & R_n \\ R_{n+r-2} & R_{n+r-3} & \cdots & R_n & R_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ R_{n+1} & R_n & & R_{n-r+3} & R_{n-r+2} \\ R_n & R_{n-1} & & R_{n-r+2} & R_{n-r+1} \end{vmatrix} = (-1)^{(r-1)n+[(r-1)/2]} \tag{1.3}$$

which is an extension of the Simson formula (identity) for the simplest case $r = 2$ for Fibonacci numbers.

Carrying these numbers R_n through to coordinate notation (writing $x_1 = R_n$, $x_2 = R_{n+1}$, $x_3 = R_{n+2}$, $\ldots$, $x_r = R_{n+r-1}$), the author [4] showed that (1.3) could be interpreted as one or more hypersurfaces in Euclidean space of r dimensions (the number of hypersurface loci depending on n). The cases $r = 2, 3, 4$ were delineated in a little detail ([3], [4]).

It is now proposed to extend the results in [3] and [4] to the case of a Lucas sequence $\{S_n\}$ or order r, i.e., to construct a determinant analogous to (1.3) and to interpret it geometrically as a locus in r-space.

From experience, we should expect the algebraic aspects of $\{S_n\}$ to resemble those of $\{R_n\}$. Nevertheless, there are sufficient variations from the Fibonacci case to make the algebraic maneuvers, which constitute the main part of this article, a challenging and absorbing exercise.

Because of complications associated with the fact that S_0 [to be defined in (2.1)] is nonzero, whereas $R_0 = 0$, the method used by Hoggatt and Bicknell [1] for $\{R_n\}$ is not applied here for $\{S_n\}$. However, our method is applicable to $\{R_n\}$, as we shall see, provided we add to the definitions in [1] the injunction $R_{-(r-1)} = 1$.

Schematically, this paper consists of two parts. Part I is organized to secure results for the Lucas sequence which correspond to those for the Fibonacci sequence. On the basis of this knowledge, in Part II we briefly generalize the results for a sequence which contains the Fibonacci and Lucas (and other) sequences as special cases.

PART I

2. LUCAS SEQUENCE OF ORDER r

Define $\{S_n\}$, the *Lucas sequence of order* r ($\geqslant 2$) by

$$S_{n+r} = S_{n+r-1} + S_{n+r-2} + \cdots + S_n, \quad S_0 = 2,\ S_1 = 1, \tag{2.1}$$

with other initial conditions

$$\begin{cases} S_{-1} = S_{-2} = \cdots = S_{-(r-2)} = 0 \\ S_{-(r-1)} = -1. \end{cases} \tag{2.2}$$

Simplest special cases of $\{S_n\}$ occur as follows:

$$L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2,\ L_1 = 1,\ L_{-1} = -1; \tag{2.3}$$

$$M_{n+3} = M_{n+2} + M_{n+1} + M_n, \quad M_0 = 2,\ M_1 = 1,\ M_{-1} = 0,\ M_{-2} = -1; \tag{2.4}$$

$$N_{n+4} = N_{n+3} + N_{n+2} + N_{n+1} + N_n, \quad N_0 = 2,\ N_1 = 1,\ N_{-2} = N_{-1} = 0,\ N_{-3} = -1. \tag{2.5}$$

The first few numbers of these sequences are:

$$\begin{cases} L_0 & L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & \cdots \\ 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & 76 & 123 & \cdots \end{cases} \tag{2.3$'$}$$

$$\begin{cases} M_0 & M_1 & M_2 & M_3 & M_4 & M_5 & M_6 & M_7 & M_8 & M_9 & M_{10} & \cdots \\ 2 & 1 & 3 & 6 & 10 & 19 & 35 & 64 & 118 & 217 & 399 & \cdots \end{cases} \tag{2.4$'$}$$

$$\begin{cases} N_0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & N_7 & N_8 & N_9 & N_{10} & \cdots \\ 2 & 1 & 3 & 6 & 12 & 22 & 43 & 83 & 160 & 308 & 594 & \cdots \end{cases} \tag{2.5$'$}$$

The determinant of order r (which we may here call the *Lucas determinant of order* r, corresponding to that in (1.3) for the Fibonacci sequence of order r) is

$$\Delta_r = \begin{vmatrix} S_{n+r-1} & S_{n+r-2} & \cdots & S_{n+1} & S_n \\ S_{n+r-2} & S_{n+r-3} & \cdots & S_n & S_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{n+1} & S_n & \cdots & S_{n-r+3} & S_{n-r+2} \\ S_n & S_{n-1} & \cdots & S_{n-r+2} & S_{n-r+1} \end{vmatrix}. \tag{2.6}$$

Notice the cyclical nature of the elements in the columns of Δ_r. Consequently, there is symmetry about the leading diagonal of Δ_r. Both of these properties for $\{S_n\}$ are also features of the Fibonacci sequence $\{R_n\}$.

Special notation: We use the symbol r_i' to mean the operation of subtracting from row i the sum of all the other rows, in a determinant of arbitrary order. An operation such as r_i' may be called a *basic operation*. Clearly, r_i' utilizes the defining recurrence (2.1) with (2.2).

It is now necessary to introduce the concept of a basic Lucas determinant.

3. BASIC LUCAS DETERMINANTS

Let us define the *basic Lucas determinant of order* r, δ_r, as

$$\delta_r = \begin{vmatrix} S_{r-1} & S_{r-2} & S_{r-3} & S_{r-4} & \cdots & S_2 & S_1 & S_0 \\ S_{r-2} & S_{r-3} & S_{r-4} & & & & S_0 & 0 \\ S_{r-3} & S_{r-4} & & & & & & 0 \\ S_{r-4} & & & & & & & 0 \\ \vdots & & & & & & & \vdots \\ S_2 & & & & & & & 0 \\ S_1 & S_0 & & & & \mathbf{0} & & 0 \\ S_0 & 0 & 0 & 0 & \cdots & & & -1 \end{vmatrix}. \tag{3.1}$$

All elements in a given upward slanting line are the same, e.g., all the elements in the reverse (upward) diagonal are S_0 $(= 2)$. Except for the element $(= -1)$ in the bottom right-hand corner, all the elements below the reverse diagonal are zero.

Observe the cyclical nature of elements in the columns, remembering initial conditions (2.2) applying to symbols with negative suffixes.

Of course, (3.1) is only the special case of (2.6) when $n = 0$.

Concerning basic Lucas determinants, we now prove the following theorem (a determinantal recurrence relation).

Theorem: $\delta_r = (-1)^{[r/2]}2^r + (-1)^{r-1}\delta_{r-1}$. (3.2)

Proof: Expand δ_r in (3.1) along the bottom row to obtain

$$\delta_r = (-1)^{[r/2]}2^r - \begin{vmatrix} S_{r-1} & S_{r-2} & S_{r-3} & \cdots & S_2 & S_1 \\ S_{r-2} & S_{r-3} & & & S_1 & S_0 \\ S_{r-3} & & & & & 0 \\ \vdots & & & & & \vdots \\ S_2 & S_1 & & & \mathbf{0} & \\ S_1 & S_0 & 0 & \cdots & & 0 \end{vmatrix} \qquad (S_1 = 1,\ S_0 = 2)$$

$$= (-1)^{[r/2]}2^r - \begin{vmatrix} 2 & 0 & 0 & \cdots & 0 & -1 \\ S_{r-2} & S_{r-3} & S_{r-4} & & 1 & 2 \\ S_{r-3} & S_{r-4} & & & 2 & 0 \\ \hdashline S_2 & 1 & 0 & & & \\ 1 & 2 & 0 & \cdots & \mathbf{0} & 0 \end{vmatrix} \qquad \text{by } r_1'$$

$= (-1)^{[r/2]}2^r - (-1)^{r-2}\delta_{r-1}$ after $r - 2$ cyclical row interchanges

$= (-1)^{[r/2]}2^r + (-1)^{r-1}\delta_{r-1}$.

Thus, we have, for $r \geqslant 2$,

$$[r = 2] \quad \delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -2^2 - 1 = -5 = \underline{\underline{-(2^3 - 3)}} \tag{3.3}$$

$$[r = 3] \quad \delta_3 = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & -1 \end{vmatrix} = -2^3 + \delta_2 = -2^3 - 2^2 - 1 = -13 = \underline{\underline{-(2^4 - 3)}} \tag{3.4}$$

$$[r = 4] \quad \delta_4 = \begin{vmatrix} 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & -1 \end{vmatrix} = 2^4 - \delta_3 = 2^4 + 2^3 + 2^2 + 1 = 29 = \underline{\underline{2^5 - 3}} \tag{3.5}$$

$$[r = 5] \quad \delta_5 = \begin{vmatrix} 12 & 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 \end{vmatrix} = 2^5 + \delta_4 = 2^5 + 2^4 + 2^3 + 2^2 + 1 = 61 = \underline{\underline{2^6 - 3}} \tag{3.6}$$

$$[r = 6] \quad \delta_6 = \begin{vmatrix} 24 & 12 & 6 & 3 & 1 & 2 \\ 12 & 6 & 3 & 1 & 2 & 0 \\ 6 & 3 & 1 & 2 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 \end{vmatrix} = -2^6 - \delta_5 = -2^6 - 2^5 - 2^4 - 2^3 - 2^2 - 1 = -125 = \underline{\underline{-(2^7 - 3)}} \tag{3.7}$$

and so on.

The emerging summation pattern by which the δ_r may be evaluated is clearly discernible. Notice that the term 2^1 (i.e., 2) does not occur in any δ_r summation.

However, before establishing the value of δ_r, we display the following tabulated information, for all possible values of r:

	$r = 4k$	$r = 4k + 1$	$r = 4k + 2$	$r = 4k + 3$
$[r/2]$	$2k$	$2k$	$2k + 1$	$2k + 1$
$(r - 1) + \left[\frac{r-1}{2}\right]$	$6k - 2$	$6k$	$6k + 1$	$6k + 3$
	even	even	odd	odd

(3.8)

From (3.8), we deduce

$$(-1)^{[r/2]} = (-1)^{r-1+[(r-1)/2]}. \tag{3.9}$$

Invoking this result and applying (3.2) repeatedly, we may now calculate the value of δ_r.

Theorem: $\delta_r = (-1)^{[r/2]}(2^{r+1} - 3)$. (3.10)

Proof:
$$\begin{aligned}
\delta_r &= (-1)^{[r/2]}2^r + (-1)^{r-1}\{(-1)^{[(r-1)/2]}2^{r-1} + (-1)^{r-2}\delta_{r-2}\} && \text{by (3.2)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1}\} - \delta_{r-2} \;\ldots\ldots\; (\alpha) && \text{by (3.9)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1}\} - (-1)^{[(r-2)/2]}\{2^{r-2} + 2^{r-3}\} + \delta_{r-4} && \text{by } (\alpha)\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + 2^{r-3}\} + \delta_{r-4}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + \cdots + 2^3 + 2^2 + 1\} && \text{ultimately, by (3.3) or by (3.4)}\\
&= (-1)^{[r/2]}\{2^r + 2^{r-1} + 2^{r-2} + \cdots + 2^3 + 2^2 + 2 + 1 - 2\}\\
&= (-1)^{[r/2]}\left\{\frac{2(1-2^r)}{1-2} - 1\right\} && \text{summing the finite geometric progression}\\
&= (-1)^{[r/2]}(2^{r+1} - 3).
\end{aligned}$$

Checking back shows that the special cases of δ_r listed in (3.3)-(3.7) have values in accord with (3.10), as expected.

4. EVALUATION OF LUCAS DETERMINANTS

Next, we show that [cf. (2.6), (3.1)]

$$\Delta_r = \pm\delta_r.$$

To illustrate the ideas involved in the proof we shall give for this connection between Δ_r and δ_r, suppose we take $r = 5$, $n = 3$, i.e., r is *odd*. This implies that we are dealing with the integer sequence

$$\left\{\begin{matrix} S_{-4} & S_{-3} & S_{-2} & S_{-1} & S_0 & S_1 & S_2 & S_3 & S_4 & S_5 & S_6 & S_7 & S_8 & \cdots\\ -1 & 0 & 0 & 0 & 2 & 1 & 3 & 6 & 12 & 24 & 46 & 91 & 179 & \cdots \end{matrix}\right. \tag{4.1}$$

Perform the basic operations r_1', r_2', r_3' successively on the determinant Δ_5 when $n = 3$ to derive:

$$\underbrace{\begin{vmatrix} 91 & 46 & 24 & 12 & 6\\ 46 & 24 & 12 & 6 & 3\\ 24 & 12 & 6 & 3 & 1\\ 12 & 6 & 3 & 1 & 2\\ 6 & 3 & 1 & 2 & 0 \end{vmatrix}}_{\Delta_5} = \underbrace{\begin{vmatrix} 3 & 1 & 2 & 0 & 0\\ 1 & 2 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & -1\\ 12 & 6 & 3 & 1 & 2\\ 6 & 3 & 1 & 2 & 0 \end{vmatrix}}_{\delta_5^*} = \underbrace{\begin{vmatrix} 12 & 6 & 3 & 1 & 2\\ 6 & 3 & 1 & 2 & 0\\ 3 & 1 & 2 & 0 & 0\\ 1 & 2 & 0 & 0 & 0\\ 2 & 0 & 0 & 0 & -1 \end{vmatrix}}_{\delta_5} \; [= 61, \text{ see } (3.6)] \tag{4.2}$$

In Δ_5, the leading term 91 ($= S_7$) is reduced to the leading term 12 ($= S_4$) in δ_5 by the $7 - 4 = 3$ ($= n$) basic operations specified. Because of the cyclical nature of Δ_5, these basic operations act to produce a determinant $\delta_5^* = \Delta_5$ whose rows are the permutation

$$\begin{bmatrix} 12 & 6 & 3 & 1 & 2\\ 3 & 1 & 2 & 12 & 6 \end{bmatrix}$$

of the rows of δ_5. Due to the fact that r is odd, this cyclic permutation is even. Expressed otherwise, δ_5^* is transformed to δ_5 by an even number of row interchanges, so the sign associated with δ_5 is +, i.e., $\delta_5^* = \delta_5$. Hence, $\Delta_5 = \delta_5$.

If $n > 5$, the cyclic process of basic operations is continued until the basic determinant δ_5 is reached. Obviously, the actual value of n is irrelevant.

The reasoning inherent in the case $r = 5$ applies equally well to the case when r is arbitrarily odd. Consequently, for r odd, $\Delta_r = \delta_r$ always.

If r is *even*, the situation is a little more complicated.

For illustrative purposes, let us examine the case $r = 4$. Substituting the numbers in (2.5)′ into (2.6) when $n = 3, 4, 5, 6$ in turn, we readily calculate that Δ_4 is reduced to the four δ_4^* whose rows are respectively the permutations

$$\begin{bmatrix} 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 6 & 3 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 2 & 6 & 3 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 3 & 1 & 2 \\ 1 & 2 & 6 & 3 \end{bmatrix}$$

of the rows of δ_4 (and this is true here for $n = 4k - 1, 4k, 4k + 1, 4k + 2$, respectively ($k = 1, 2, \ldots$).

As these permutations are successively odd, even, odd, even, it follows that $\delta_4^* = -\delta_4, \delta_4, -\delta_4, \delta_4$ in turn. Thus, $\Delta_4 = \pm\delta_4$, depending on n, namely, $\Delta_4 = \delta_4$ when n is even, while $\Delta_4 = -\delta_4$ when n is odd.

Armed with this knowledge, we can now attack the general problem, i.e., when r and n are arbitrary integers.

First, we establish the following result:

Theorem: $\Delta_r = (-1)^m \delta_r$, where $m = n'(r - n')$, $n' \equiv n \bmod r$. (4.3)

Proof: In (2.6), the leading term S_{n+r-1} in Δ_r is diminished to the leading term S_r in δ_r by $n + r - 1 - (r - 1) = n$ basic operations $r_1', r_2', \ldots, r_n'$ which produce the determinant δ_r^*. Simultaneously, S_{r-1} drops n places in the first column of δ_r^*.

To restore the cyclical order in the first column of δ_r^* to the basic cyclical order of the first column in δ_r, beginning with S_{r-1}, it is necessary to effect $n(r - n)$ interchanges of sign to account for the $r - n$ terms below and including S_{r-1}, and the n terms above S_{r-1}.

When $n > r$, we reduce n mod r.

Each interchange accounts for a change of sign in the value of the determinant.

When r is odd, the product $n(r - n)$ is always even, no matter what the value of n is.

But when r is even, the product $n(r - n)$ is odd if n is odd, and even when n is even. [That is, when r is a given odd number there is only one value of Δ_r, whereas for a given even r there are two values of Δ_r depending on the value of n.]

Thus,

$$\Delta_r = (-1)^m \delta_r, \text{ where } m = n'(r - n'),\ n' \equiv n \bmod r.$$

Combining (3.10) and (4.3), we have the following theorem as an immediate deduction.

Theorem: $\Delta_r = (-1)^{m+[r/2]}(2^{r+1} - 3)$, where $m = n'(r - n')$, $n' \equiv n \bmod r$. (4.4)

For example,

$$\text{when } r = 5,\ n = 3:\quad \Delta_4 = (-1)^{3\times 2+2}(2^6 - 3) \quad \text{from (4.4)}$$
$$= +61$$

in conformity with (4.2) and (3.6).

On the other hand,

$$\text{when } r = 5,\ n = 4: \quad \Delta_4 = (-1)^{1\times 3+2}(2^5 - 3) \qquad \text{from (4.4)}$$
$$= -29$$

as we have seen in the discussion preceding (4.3).

Applying (4.4) to a random choice $r = 6$, $n = 8$ (but not so random that the computations are unmanageable!), we discover on substitution that

$$\Delta_6 = (-1)^{2\times 4+3}(2^7 - 3) = -125,$$

as may be verified by direct calculation.

Our result (4.4) for a Lucas sequence of order r should be compared with the Hoggatt-Bicknell result (1.3) for the corresponding Fibonacci case.

5. HYPERSURFACES FOR THE LUCAS SEQUENCES

Geometrical interpretations can now be given to the identity (4.4) and its specializations for small values of r. The reader is referred to [3] and [4] for details of the geometry relating to Simson-type identities for Fibonacci sequences of order r.

As this corresponding work for Simson-type identities for Lucas sequences of order r parallels the results in [3] and [4], we will content ourselves here with a fairly brief statement of the main ideas.

Write $x_1 = S_n$, $x_2 = S_{n+1}$, $\ldots$, $x_r = S_{n+r-1}$. Represent a point in r-dimensional Euclidean space by Cartesian coordinates $(x_1, x_2, \ldots, x_r)$.

Then, we interpret (4.4) as the equation of a locus of points in r-space which has maximum dimension $r - 1$ in the containing space. Such a locus is called a *hypersurface*.

Each of the loci given by the simple linear equations $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_r = 0$ is a "flat" (linear) space of dimension $r - 1$, and is called a (coordinate) *hyperplane*.

Hypersurfaces of the simplest kind occur for small values of r. In accord with our theory, there will be one hypersurface when r is odd, and two when r is even.

Examples of hypersurfaces for $\{S_n\}$ are:

$$r = 2 \text{ (conic)}: \quad x_1^2 + x_1x_2 - x_2^2 = 5(-1)^n \tag{5.1}$$

$$r = 3 \text{ (cubic surface)}: \quad x_1^3 + 2x_2^2 + x_3^2 + 2x_1^2x_2 + 2x_1x_2^2 - 2x_2x_3^2 + x_1^2x_3 - x_1x_3^2 - 2x_1x_2x_3 = -13. \tag{5.2}$$

In passing, we note that (5.1) expresses the well-known Simson-type identity for Lucas sequences of order 2, namely,

$$L_{n+1}L_{n-1} - L_n^2 = 5(-1)^{n+1}. \tag{5.1$'$}$$

Moreover, the matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix},$$

whose determinant δ_2 is associated with identity (5.1)$'$, has several interesting geometrical interpretations (relating to: angle-bisection, reflection, vector mapping). (See Hoggatt and Ruggles [2].)

Observe that if we replace x_1, x_2, x_3 by x, y, z, respectively, in (5.1) and (5.2), we obtain equations whose forms, except for the numbers on the right-hand sides, are identical to those given in [3] and [4]. However, this formal structure camouflages the fact that the corresponding equations of the conics and cubic surfaces, for Fibonacci and Lucas sequences, are satisfied by different sets of numbers.

Carrying further our comparison with the results for Lucas and Fibonacci sequences, we obtain, in the case $r = 4$, a nasty equation (refer to [4]) for a quartic hypersurface in four-dimensional Euclidean space. And so on for hypersurfaces in higher dimensions.

Sections of these loci by coordinate hyperplanes yield plane curves of various orders (cubics, quartics, quintics, sextics, and, generally, curves of order r). Refer here also to [4].

This completes our summarized outline of the geometrical consequences of of the determinantal identity (4.4) for Lucas sequences paralleling those for the Fibonacci sequences.

With the notions of Part I in mind, we are in a position to examine closely a more general sequence of order r which has the Fibonacci and Lucas sequences of order r as special cases.

PART II

Only a brief outline of the ensuing generalization, which parallels the information in Part I, will be given.

6. A GENERALIZED SEQUENCE OF ORDER r

Let us now introduce the generalized sequence $\{H_n\}$ defined by the recurrence relation

$$H_{n+r} = H_{n+r-1} + H_{n+r-2} + \cdots + H_n, \quad H_0 = a, \ H_1 = b, \tag{6.1}$$

with further initial conditions

$$\begin{cases} H_{-1} = H_{-2} = \cdots = H_{-(r-2)} = 0 \\ H_{-(r-1)} = b - a. \end{cases} \tag{6.2}$$

Interested readers might wish to write out the first few terms of these sequences for different values of r. For example, H takes on, in turn, the values $5a + 8b$, $11a + 13b$, $14a + 15b$, $15a + 16b$, and $16a + 16b$ for successive values $r = 2, 3, 4, 5$, and 6.

As in (2.6), the *generalized determinant of order* r is defined to be

$$D_r = \begin{vmatrix} H_{n+r-1} & H_{n+r-2} & \cdots & H_{n+1} & H_n \\ H_{n+r-2} & H_{n+r-3} & \cdots & H_n & H_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ H_{n+1} & H_n & \cdots & H_{n-r+3} & H_{n-r+2} \\ H_n & H_{n-1} & \cdots & H_{n-r+2} & H_{n-r+1} \end{vmatrix} \tag{6.3}$$

Evaluation of D_r is the object of Part II.

Define, as in (3.1), the *basic generalized determinant of order* r, d_r, to be obtained from (6.3) when $n = 0$. That is,

$$d_r = (D_r)_{n=0}. \tag{6.4}$$

Then, the following simplest basic generalized determinants may be readily calculated by expanding along the bottom row:

$$[r = 2] \quad d_2 = -a^2 + (b - a)b \tag{6.5}$$

$$[r = 3] \quad d_3 = -a^3 - (b - a)d_2 = -a^3 + (b - a)a^2 - (b - a)^2 b \tag{6.6}$$

$$\begin{aligned}[r = 4] \quad d_4 &= a^4 + (b - a)d_3 \\ &= a^4 - (b - a)a^3 + (b - a)^2 b^2 - (b - a)^3 b\end{aligned} \tag{6.7}$$

$$\begin{aligned}[r = 5] \quad d_5 &= a^5 - (b - a)d_4 \\ &= a^5 - (b - a)a^4 + (b - a)^2 b^3 - (b - a)^3 a^2 + (b - a)^4 b\end{aligned} \tag{6.8}$$

$$\begin{aligned}[r = 6] \quad d_6 &= -a^6 + (b - a)d_5 \\ &= -a^6 + (b - a)a^5 - (b - a)^2 a^4 + (b - a)^3 a^3 \\ &\qquad - (b - a)^4 a^2 + (b - a)^5 b\end{aligned} \tag{6.9}$$

and so on.

A developing pattern is clearly discernible.

Calculation of d_r follows the method employed in (3.2).

Although only an outline of the theory in Part II is being offered, it is generally desirable for clarity of exposition to exhibit the main points of the calculation of d_r in a little detail, even at the risk of some possibly superfluous documentation.

Theorem: $d_r = (-1)^{[r/2]}\{a^{r+1} + (-1)^{r-1}(a + b)(b - a)^r\}/b.$ (6.10)

Proof: Expand d_r in (6.3) and (6.4) along the last row to obtain

$$\begin{aligned}
d_r &= (-1)^{[r/2]}a^r - (b - a)(-1)^{r-1}d_{r-1} \;\ldots\ldots\ldots\; \text{(i)} \\
&= (-1)^{[r/2]}\{a^r - (b - a)a^{r-1}\} - (b - a)^2 d_{r-2} \qquad \text{by (i), (3.9)} \\
&= (-1)^{[r/2]}\{a^r - (b - a)a^{r-1} + (b - a)^2 a^{r-2} - (b - a)^3 a^{r-3} + \cdots \\
&\qquad + (-1)^{r-2}(b - a)^{r-2}a^2 + (-1)^{r-1}(b - a)^{r-1}b \\
&\qquad + [(-1)^{r-1}(b - a)^{r-1}a - (-1)^{r-1}(b - a)^{r-1}a]\} \\
&= (-1)^{[r/2]}\{a^r(1 - (b - a)a^{-1} + (b - a)^2 a^{-2} - (b - a)^3 a^{-3} + \cdots \\
&\qquad + (-1)^r(b - a)^{r-2}a^{-(r-2)} + (-1)^{r-1}(b - a)^{r-1}a^{-(r-1)}) \\
&\qquad + (-1)^{r-1}(b - a)^r\} \\
&= (-1)^{[r/2]}\left\{a^r\frac{[1 - (-(b - a)a^{-1})^r]}{1 - (-(b - a)a^{-1})} + (-1)^{r-1}(b - a)^r\right\} \\
&- (-1)^{[r/2]}\{a^{r+1} + (-1)^{r-1}(a + b)(b - a)^r\}/b.
\end{aligned}$$

Repeated use of (i) has been made in the proof. Also, the summation formula for a finite geometric progression has been invoked.

Applying next the arguments used in the evaluation of the Lucas determinant or order r, Δ_r, we easily have

Theorem: $D_r = (-1)^m d_r = (-1)^{m+[r/2]}\{a^{r+1} + (-1)^{r-1}(a+b)(b-a)^r\}/b,$ (6.11)

where $\begin{cases} m = n'(r-n') \\ n' \equiv n \mod r. \end{cases}$

Proof: As for (4.3).

For the Lucas sequence of order r, $\{S_n\}$,

$$a = 2,\ b = 1\quad (\text{so } a + b = 3,\ b - a = -1).$$

It is easy to verify that, in this case,

$$d_r = \delta_r,\quad D_r = \Delta_r.$$

[Cf. (3.10), (4.4).]

Coming now to the case of the Fibonacci sequence of order r, $\{R_n\}$, we have

$$a = 0,\ b = 1\quad (\text{so } a + b = 1,\ b - a = 1).$$

It is important to note that, for our theory to be used for $\{R_n\}$, the terms of $\{R_n\}$ with negative suffixes have to be extended by one term in the definition (1.1), (1.2) given by Hoggatt and Bicknell [1], namely,

$$R_{-(r-1)} = 1. \tag{6.12}$$

Augmenting $\{R_n\}$ by this single element enables us to construct *basic Fibonacci determinants* of order r, ∇_r, for $\{R_n\}$ derived from (1.3) analogously to those for $\{S_n\}$ from (2.6). Computation yields

$$\nabla_2 = 1,\ \nabla_3 = -1,\ \nabla_4 = -1,\ \nabla_5 = 1,\ \nabla_6 = 1.$$

To give some appreciation of the appearance of the ∇_r, choose $r = 6$, so

$$\nabla_6 = \begin{vmatrix} 8 & 4 & 2 & 1 & 1 & 0 \\ 4 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \quad (= 1) \tag{6.13}$$

which is rather simpler than the corresponding form (3.7) for δ_6.

Putting $a = 0$, $b = 1$ in (6.10), we have, with the aid of (3.9)

$$d_r = (-1)^{[r/2]+r-1} = (-1)^{[(r-1)/2]} = \nabla_r.$$

Now it may be shown that $m + [(r-1)/2]$, the power of -1 in (6.11), and $(r-1)n + [(r-1)/2]$, the corrected power of -1 in the Hoggatt-Bicknell [1] evaluation in (1.3), are both even or both odd. That is

$$(-1)^{m+[r/2]} = (-1)^{(r-1)n+[(r-1)/2]}. \tag{6.14}$$

Thus, $a = 0$, $b = 1$ in (6.11) with (3.9) give

$$D_r = (-1)^{m+[(r-1)/2]} = (-1)^{(r-1)n+[(r-1)/2]} = \nabla_r,$$

where ∇_r is the symbol to represent the Hoggatt-Bicknell determinant (1.3).

Suppose we check for $\{R_n\}$ when $r = 6$, $n = 3$, i.e., we are dealing with the sequence

$$\begin{cases} R_{-5} & R_{-4} & R_{-3} & R_{-2} & R_{-1} & R_0 & R_1 & R_2 & R_3 & R_4 & R_5 & R_6 & R_7 & R_8 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 8 & 16 & 32 & 63 & \ldots \end{cases} \tag{6.15}$$

Then

$$\begin{aligned} (n = 3): \quad D_6 &= (-1)^{3\times 3+2} = -1 && \text{by (6.11), (3.9)} \\ &= (-1)^{5\times 3+2} = -1 && \text{by (1.3)} \\ &= -\nabla_6 \qquad\; = -1 && \text{on direct calculation.} \end{aligned}$$

Geometrical considerations similar to those in [4] and in Part I of this article are now applicable to the general case of $\{H_n\}$ when a and b are unspecified, and also to the multifarious special cases of $\{H_n\}$ occurring when a and b are given particular values.

But we do not proceed *ad infinitum, ad nauseam* by discussing other classes of sequences. Unsatiated readers, if such there be, may indulge to surfeit in such an algebraic geometry orgy.

One further generalization might be contemplated if, in (6.1), we were to associate with each H_{n+r-j} $(j = 1, 2, \ldots, r)$ a nonzero, nonunity factor p_j. However, the algebra involved makes this a daunting prospect.

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4. A. F. Horadam. "Hypersurfaces Associated with Simson Formula Analogues." *The Fibonacci Quarterly* **24**, no. 3 (1986):221-226.
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*To the list of references, we take the liberty of adding the monumental text [5] by Room, though its subject is determinantal loci in projective, not Euclidean, spaces.

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