A FURTHER NOTE ON PASCAL GRAPHS

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1. INTRODUCTION

In a recent paper Deo and Quinn [1], in their search for a class of graphs to be used as computer networks, introduced Pascal graphs that are constructed using Pascal's triangle modulo 2 [3]. They derived a number of useful results for Pascal graphs and Pascal matrices and, in the conclusion, they made certain interesting conjectures. The objective of the present note is to find an exact expression for the number of edges in Pascal graphs of different orders and to settle one of the conjectures made in [1].

We have used standard graph theoretic terms [4], [5] in this paper, and the reader is assumed to be familiar with [1].

2. BASIC CONCEPTS

Definition 1: A Pascal matrix PM_n of order n is defined to be an $n \times n$ symmetric binary matrix where the main diagonal entries are all 0's and the lower triangular part of the matrix consists of the first $(n - 1)$ rows of Pascal's triangle modulo 2. $PM_n(i, j)$ denotes the $(i, j)^{\text{th}}$ element of PM_n. A Pascal graph PG_n having n vertices is a graph corresponding to the adjacency matrix PM_n.

Remark: This definition of a Pascal matrix is the same as in [1] in contrast to that in [2].

Definition 2: The generator polynomial of the m^{th} row, $m \geqslant 1$, of a Pascal matrix PM_n of any fixed order $n \geqslant m$ is defined to be a polynomial $f_m(x)$ with binary coefficients such that $PM_n(m, j)$ is given by the coefficient of x^{j-1} in $f_m(x)$, $1 \leqslant j \leqslant n$.

Since $PM_n(m, m) = 0$ by definition, we can write, for a Pascal matrix PM_n, $n \geqslant m$,

$$f_m(x) = \begin{cases} g_m(x) + x^m h_m(x), & \text{for } n > m \\ g_m(x), & \text{for } n = m, \end{cases}$$

where $g_m(x)$ and $h_m(x)$ are the generator polynomials of the lower and the upper triangular parts, respectively, of the m^{th} row in PM_n. By definition, $g_m(x)$ applies only for $m \geqslant 2$.

Definition 3: The B-sequence of a positive integer n is defined as the strictly decreasing sequence $B(n) = (p_1, p_2, \ldots, p_{L_n})$ of L_n nonnegative integers such that

$$n = \sum_{j=1}^{L_n} 2^{p_j},$$

where L_n is the length of the sequence.

Remarks: (1) The B-sequence of any positive integer n gives the positions of 1's in the binary representation of n in decreasing order.

(2) The B-sequence of zero is defined to be a null sequence.

Lemma 1: For any Pascal matrix PM_n with $n \geqslant m$,

(a) For $m \geqslant 2$, $g_m(x) = \prod_{j \in B(m-2)} (1 + x^{2^j})$.

(b) For $m \geqslant 1$, $h_m(x) = \prod_{\substack{j \geqslant 0 \\ j \notin B(m-1)}} (1 + x^{2^j})$.

Proof:

(a) From the definitions of a Pascal matrix and $g_m(x)$, it is apparent that $g_m(x) = (1 + x)^{m-2}$, with the coefficients computed in the modulo 2 field, from which the proof follows.

(b) Since PM_n is symmetric, $h_m(x)$ will contain x^k as a nonzero term iff $g_{m+k+1}(x)$ contains x^{m-1} as a nonzero term, $k \geqslant 0$. This is possible if and only if $B(m+k-1)$ contains $B(m-1)$ as a subsequence, i.e., when there is no element common to both $B(k)$ and $B(m-1)$. Hence the claim.

Example: In a Pascal matrix of order $n = 30$,

$$f_{13}(x) = (1 + x)(1 + x^2)(1 + x^8) + x^{13}(1 + x)(1 + x^2)(1 + x^{16}),$$

$$f_{20}(x) = (1 + x^2)(1 + x^{16}) + x^{20}(1 + x^4)(1 + x^8).$$

Remark: For any m, $m \geqslant 2$, $(1 + x)$ is a factor of $g_m(x)$ iff $(1 + x)$ is also a factor of $h_m(x)$, since $B(n)$, $n > 0$, can contain 0 only when n is odd.

Definition 4: The m^{th} row of PM_n will be called the p^{th} instance of all 1's in the lower triangle if $m = 2^p + 1$, $p \geqslant 1$.

3. NUMBER OF EDGES IN PASCAL GRAPHS

Let $e(n)$ denote the number of edges in PG_n. Deo and Quinn [1] showed that

$$e(n) \leqslant \lfloor (n-1)^{\log_2 3} \rfloor.$$

In this section we find an exact expression for $e(n)$.

Lemma 2: In a Pascal graph PG_n, where

$$n = (2^p + 1) + i,$$

for some nonnegative integer p and $1 \leqslant i \leqslant 2^p$, the degree $d(n)$ of the n^{th} vertex in PG_n is given by

$$d(n) = 2d(i + 1),$$

where $d(i + 1)$ denotes the degree of the $(i + 1)^{\text{st}}$ vertex in PG_{i+1}.

Proof: In PM_n, the n^{th} row has only its lower triangular part and so does the $(i + 1)^{\text{th}}$ row in PM_{i+1}. Hence, in PM_n,

$$f_n(x) = g_n(x) = (1 + x)^{n-2} = (1 + x)^{2^p} \cdot (1 + x)^{i-1}.$$

Since the coefficients of the polynomials are computed in a modulo 2 field, we get

$$g_n(x) = (1 + x^{2^p}) \cdot g_{i+1}(x).$$

Therefore, since $i \leqslant 2^p$, the number of nonzero terms in $g_n(x)$ is twice that in $g_{i+1}(x)$. Hence $d(n) = 2d(i + 1)$. Q.E.D.

If the n^{th} row of PM_n corresponds to the p^{th} instance $(p \geqslant 1)$ of all 1's in its lower triangular part, i.e., if $n = 2^p + 1$, then we also denote the number of edges in PG_n by $E(p)$, i.e., $E(p) = e(2^p + 1)$.

Lemma 3: $E(p) = 3^p$.

Proof:

$$\begin{aligned} E(p) &= \text{Number of edges in } PG \text{ of order } (2^{p-1} + 1) \\ &\quad + \text{Number of edges added due to addition of extra } 2^{p-1} \text{ vertices} \\ &= E(p - 1) + 2E(p - 1) \qquad [\text{by Lemma 2}] \\ &= 3E(p - 1) \\ &= 3^2E(p - 2) = \cdots = 3^{p-1}E(1). \end{aligned}$$

Now $E(1)$ corresponds to the number of edges in PG_3, which is 3. Hence, we get

$$E(p) = 3^p. \quad \text{Q.E.D.}$$

Theorem 1: The number of edges in PG_n $(n > 1)$ is given by

$$e(n) = \sum_{j=1}^{L_{n-1}} 2^{j-1} \cdot 3^{p_j},$$

where $(p_1, p_2, \ldots, p_{L_{n-1}})$ is the B-sequence $B(n - 1)$ of length L_{n-1}.

Proof: Let $n - 1 = n_1 + n_2 + \cdots + n_k$, where $k = L_{n-1}$, $n_i = 2^{p_i}$, $1 \leqslant i \leqslant k$. Hence the $(n_1 + 1)^{\text{th}}$ row of PM_n corresponds to the $p_1{}^{\text{th}}$ instance of all 1's in the lower triangle, and so by Lemmas 2 and 3,

$$\begin{aligned} e(n) &= E(p_1) + \text{extra edges due to addition of vertices } v_{n_1+2}, \ldots, v_n \\ &\qquad \text{to } PG_{n_1+1} \\ &= 3^{p_1} + 2e(n_2 + n_3 + \cdots + n_k + 1) = 3^{p_1} + 2e(n'), \end{aligned}$$

where $n' = (1 + n_2) + n_3 + \cdots + n_k$. Repeating the process, we get

$$\begin{aligned} e(n) &= 3^{p_1} + 2(3^{p_2} + 2e(n'')) \quad [\text{where } n'' = n_3 + n_4 + \cdots + n_k + 1] \\ &= \cdots \\ &= \sum_{j=1}^{k} 2^{j-1} \cdot 3^{p_j}. \quad \text{Q.E.D.} \end{aligned}$$

4. DETERMINANTS OF PASCAL MATRICES

We now settle one of the conjectures made in [1] regarding the values of n for which the determinant of PM_n will be zero.

Let us consider an integer A which is either 2 or can be expressed as

$$A = 2 + (a_1 + a_2 + \cdots + a_k),$$

where $a_1 = 4$ or 8, and $a_{i+1} = 4a_i$, $1 \leqslant i \leqslant k$.

We now define the *index set* I of A as follows:

$$I = \begin{cases} \varphi, & \text{for } A = 2 \\ \{1, 2, \ldots, k\}, & \text{otherwise} \end{cases}$$

Let the cardinality of I be denoted by t. It can be verified that for $A > 2$, we can write

$$A = 1 + \sum_{j=0}^{t} 4^j, \quad \text{for } a_1 = 4 \tag{1}$$

and

$$A = 2\sum_{j=0}^{t} 4^j, \quad \text{for } a_1 = 8. \tag{2}$$

Both (1) and (2) also apply for $A = 2$, i.e., for $t = 0$ as well. Let $a_i' = 2a_i$, for $1 \leqslant i \leqslant k$. We use an arbitrary subset $I' = \{j_1, j_2, \ldots, j_p\}$ of I to denote different integers generated from A as follows:

$$A_{j_1 j_2 \ldots j_p} = 2 + \sum_{i \in I'} a_i' + \sum_{i \in I - I'} a_i.$$

When $\{j_1, j_2, \ldots, j_p\} = \varphi$, $A_{j_1 j_2 \ldots j_p} = A$ itself.

Let $P(I)$ be the power set of I. We define the expansion set $S(A)$ of A as

$$S(A) = \left\{A_{j_1 j_2 \ldots j_p} \middle| \{j_1, j_2, \ldots, j_p\} \in P(I)\right\}.$$

Example: For $A = 2$, $S(A) = \{2\}$.

For $A = 22$, $a_1 = 4$, $a_2 = 16$, $a_1' = 8$, $a_2' = 32$,

$$A_1 = 2 + a_1' + a_2 = 26,$$
$$A_2 = 2 + a_1 + a_2' = 38,$$
$$A_{12} = 2 + a_1' + a_2' = 42,$$

and $S(A) = \{22, 26, 38, 42\}$.

The *r-distant co-expansion set* $T_r(A)$ of A is defined as

$$T_r(A) = \left\{A_{j_1 j_2 \ldots j_p} + r \middle| A_{j_1 j_2 \ldots j_p} \in S(A)\right\}.$$

We construct a set of polynomials of the form $F_{j_1 j_2 \ldots j_q}^{i_1 i_2 \ldots i_p}$, where

$$\{i_1, i_2, \ldots, i_p\} \subseteq I, \ \{j_1, j_2, \ldots, j_q\} \subseteq I, \ 1 \in \{i_1, i_2, \ldots, i_p\}$$

and

$$\{i_1, i_2, \ldots, i_p\} \cap \{j_1, j_2, \ldots, j_q\} = \varphi,$$

using the following recurrence relation:

$$F^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = F^{i_1 i_2 \cdots i_{p-1}}_{j_1 j_2 \cdots j_q} - F^{i_1 i_2 \cdots i_{p-1}}_{j_1 j_2 \cdots j_q i_p},$$

with

$$F^{1}_{j_1 j_2 \cdots j_q} = (f_{A_{j_1 j_2 \cdots j_q}} - f_{A_{j_1 j_2 \cdots j_q}+2}) - (f_{A_{1 j_1 j_2 \cdots j_q}} - f_{A_{1 j_1 j_2 \cdots j_q}+2}),$$

where the f's are the generator polynomials as given in Definition 2.

It may be noted that if $\{j_1, j_2, \ldots, j_q\} = \varphi$, then the polynomial is represented simply as $F^{i_1 i_2 \cdots i_p}$, i.e., without a subscript. Moreover, the superscript set $\{i_1, i_2, \ldots, i_p\}$ can never be empty.

Example: Let $A = 22$. Hence $I = \{1, 2\}$, $A_1 = 26$, $A_2 = 38$, $A_{12} = 42$,

$$F^1 = (f_A - f_{A+2}) - (f_{A_1} - f_{A_1+2}) = (f_{22} - f_{24}) - (f_{26} - f_{28})$$

$$F^1_2 = (f_{38} - f_{40}) - (f_{42} - f_{44})$$

$$F^{12} = F^1 - F^1_2 = (f_{22} - f_{24}) - (f_{26} - f_{28}) - [(f_{38} - f_{40}) - (f_{42} - f_{44})].$$

In particular, the polynomials of the form $F^{i_1 i_2 \cdots i_p}$ will play an important role in proving the conjecture, as we shall see later on. The recursive computation of such polynomials can be visualized easily with the help of a binary tree. Consider, for example, the computation of F^{123}, which is represented by the binary tree as shown in Figure 1. The leaf nodes represent the generator polynomials corresponding to different rows of the Pascal matrix and each of the non-leaf nodes represents the arithmetic subtraction operation. Some of the non-leaf nodes are labelled, e.g., F^1, F^1_2, F^{12}, etc. The inorder traversal [6] of the subtree rooted at any labelled non-leaf node computes the polynomial denoted by that label.

Let $[\alpha_t, \beta_t]$ be a closed interval of integers given by

$$\alpha_t = 2 + 2\sum_{j=0}^{t} 4^j, \quad \beta_t = 1 + \sum_{j=0}^{t+1} 4^j, \quad t \geqslant 0. \tag{3}$$

Theorem 2: In a Pascal matrix of order n, where n lies within the closed interval $[\alpha_t, \beta_t]$, as defined in (3), the 2^{t+1} rows denoted by the expansion set $S(A)$ and the 2-distant co-expansion set $T_2(A)$ of the integer A as given in (1) are linearly dependent, i.e., the determinant of PM_n for such values of n will be zero.

Proof:

Case 1. $t = 0$

In this case, $\alpha_0 = 4$, $\beta_0 = 6$, and $A = 2$. So $S(A) = \{2\}$ and $T_2(A) = \{4\}$. Since the order n of the Pascal matrix is limited by $\beta_0 = 6$, we write

$$f_2 = 1 + x^2(1 + x^2) \quad \text{and} \quad f_4 = (1 + x^2) + x^4.$$

So

$$f_2 - f_4 = 0.$$

Case 2. $t \geqslant 1$

To prove the linear dependence among the different rows of PM_n, it is sufficient to show that any of the polynomials of the form $F^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q}$ will be zero.

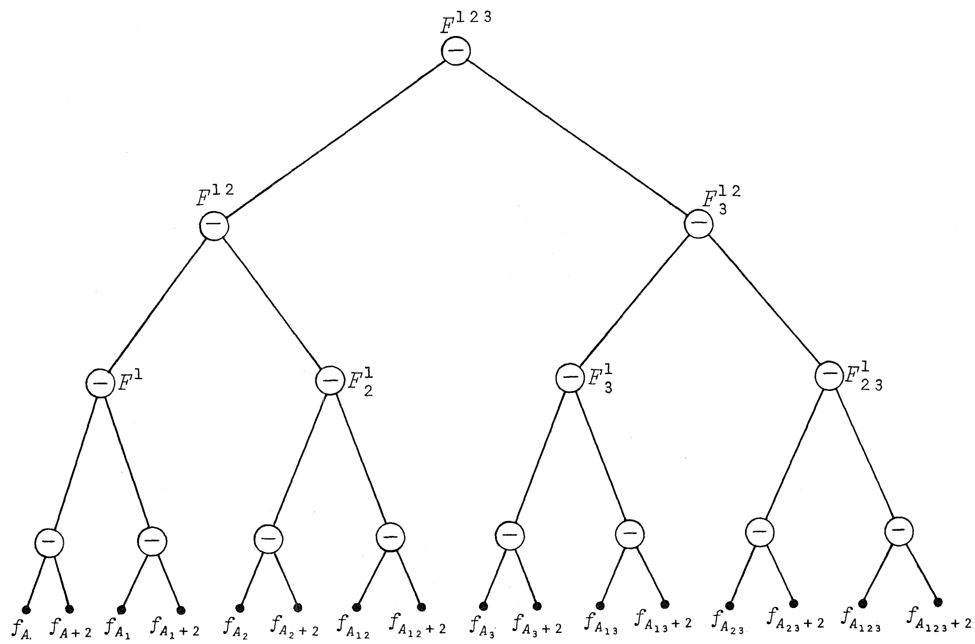

Figure 1

Since the order n of PM_n is limited by β_t, we write

$$f_A = (1 + x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 + x^2)(1 + x^{a'_1})(1 + x^{a'_2}) \cdots (1 + x^{a'_t})$$

and

$$f_{A+2} = (1 + x^2)(1 + x^{a_1}) \cdots (1 + x^{a_t}) + x^{A+2}(1 + x^{a'_1}) \cdots (1 + x^{a'_t}).$$

Hence,

$$f_A - f_{A+2} = -x^2(1 + x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 + x^{a'_1}) \cdots (1 + x^{a'_t}).$$

Similarly,

$$f_{A_1} - f_{A_1+2} = -x^2(1 + x^{a'_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^{A_1}(1 + x^{a_1})(1 + x^{a'_2}) \cdots (1 + x^{a'_t}).$$

Hence,

$$F^1 = -x^2x^{a_1}(1 - x^{a_1})(1 + x^{a_2}) \cdots (1 + x^{a_t}) + x^A(1 - x^{a_1})(1 + x^{a'_2}) \cdots (1 + x^{a'_t}).$$

Also,

$$\begin{aligned} F_2^1 &= (f_{A_2} - f_{A_2+2}) - (f_{A_{12}} - f_{A_{12}+2}) \\ &= -x^2x^{a_1}(1 - x^{a_1})(1 + x^{a'_2})(1 + x^{a_3}) \cdots (1 + x^{a_t}) \\ &\quad + x^{A_2}(1 - x^{a_1})(1 + x^{a_2})(1 + x^{a'_3}) \cdots (1 + x^{a'_t}) \end{aligned}$$

and

$$\begin{aligned} F^{12} &= F^1 - F_2^1 \\ &= -x^2x^{a_1}x^{a_2}(1 - x^{a_1})(1 - x^{a_2})(1 + x^{a_3}) \cdots (1 + x^{a_t}) \\ &\quad + x^A(1 - x^{a_1})(1 - x^{a_2})(1 + x^{a'_3}) \cdots (1 + x^{a'_t}). \end{aligned}$$

Continuing the process, we get

$$F^{123\cdots t} = -x^{2+a_1+a_2+\cdots+a_t}(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_t})$$
$$+ x^A(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_t}) = 0. \quad \text{Q.E.D.}$$

Let $[\gamma_t, \delta_t]$ be a closed interval of integers given by

$$\gamma_t = 4 + 4\sum_{j=0}^{t} 4^j, \quad \delta_t = 2\sum_{j=0}^{t+1} 4^j, \quad t \geqslant 0. \tag{4}$$

Theorem 3: In a Pascal matrix of order n, where n lies within the closed interval $[\gamma_t, \delta_t]$, as defined in (4), the 2^{t+1} rows denoted by the expansion set $S(A)$ and the 6-distant co-expansion set $T_6(A)$ of the integer A as given in (2) are linearly dependent, i.e., the determinant of PM_n for such values of n will be zero.

Proof: The proof is similar to that of Theorem 2 and is omitted here.

Remarks: (1) $\gamma_t = \beta_t + 2$ and $\alpha_{t+1} = \delta_t + 2$.

(2) For all t, $t \geqslant 0$, $[\alpha_t, \beta_t]$, and $[\gamma_t, \delta_t]$ give two series of intervals of orders of Pascal matrices having zero determinants.

(3) In a Pascal matrix PM_n, where $n = \beta_t + 1$ or $\delta_t + 1$, $t \geqslant 0$, the approach used in the proof of Theorem 2 fails to discover a set of linearly dependent rows. This can be seen as follows:

If $n = \beta_t + 1$, then we must consider terms up to $x^{\beta t}$ in the generator polynomials of the rows of PM_n. Since $\beta_t = A + 4^{t+1} = A + a_{t+1}$, $t \geqslant 0$, only the polynomial f_A will have an added term $(1 + x^{a_{t+1}})$ in its h_A-part; all other polynomials, e.g., f_{A+2}, f_{A_1}, f_{A_1+2}, ..., etc., as given in the proof of Theorem 2, will remain unaltered. Hence, $F^{123\cdots t}$ will not be zero. The case for $n = \delta\ + 1$ can be similarly verified.

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