A SOLUTION TO A TANTALIZING PROBLEM

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INTRODUCTION

In a recent paper, R. Backstrom [1] computed various sums of reciprocal Fibonacci and Lucas numbers. By a strange limit process, he also gets an estimate (to the seventh decimal place) of the sum

$$\sum_0^\infty \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha}, \text{ where } \alpha = \frac{\sqrt{5}+1}{2}$$

(here $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$). An even better estimate is the formula

$$\sum_0^\infty \frac{1}{L_{2n}+2} \approx \frac{1}{8} + \frac{1}{4\log\alpha} + \frac{\pi^2}{(\log\alpha)^2} \cdot \frac{1}{e^{\pi^2/\log\alpha} - 2},$$

which has at least thirty correct decimal places. But both these formulas are just the first terms in a very rapidly converging series, that is, a quotient of two theta functions.

This paper contains no new results. On the contrary, most of the results are approximately 150 years old, mostly due to Jacobi. The formulas for the sums of reciprocal Fibonacci and Lucas numbers are obtained by substituting $q = \alpha^{-1}$ or $q = \alpha^{-2}$ in identities valid for formal power series or for series converging for $|q| < 1$.

Probably all the results in Backstrom's paper can be obtained by specializing to $q = \alpha^{-1}$ or $q = \alpha^{-2}$ in sums of telescoping series. For example, let us look at Theorem I in [1]:

$$\sum_{n=0}^\infty \frac{1}{F_{2n+1} + F_{2r+1}} = \sqrt{5}\left(r + \frac{1}{2}\right)\Big/L_{2r+1}.$$

We have

$$\begin{aligned}\frac{1}{F_{2n+1}+F_{2r+1}} &= \frac{\sqrt{5}}{L_{2r+1}}\left(\frac{1}{1+\alpha^{-2(n+r+1)}} - \frac{1}{1+\alpha^{-2(n-r)}}\right)\\ &= \frac{\sqrt{5}}{L_{2r+1}}\left(\frac{1}{1+q^{n+r+1}} - \frac{1}{1+q^{n-r}}\right), \text{ where } q = \alpha^{-2}.\end{aligned}$$

Hence, it is sufficient to show that

$$\sum_{n=0}^\infty \left(\frac{1}{1+q^{n+r+1}} - \frac{1}{1+q^{n-r}}\right) = r + \frac{1}{2} \text{ for } 0 < |q| < 1.$$

Now,

$$\sum_{n=0}^{\infty}\left(\frac{1}{1+q^{n+r+1}}-\frac{1}{1+q^{n-r}}\right)=\sum_{\nu=N-r+1}^{N+r+1}\frac{1}{1+q^{\nu}}-\sum_{\nu=-r}^{r}\frac{1}{1+q^{\nu}}$$

$$\to 2r+1-\left(r+\frac{1}{2}\right)=r+\frac{1}{2}\quad\text{as } N\to\infty,$$

since

$$\frac{1}{1+q^{\nu}}+\frac{1}{1+q^{-\nu}}=1 \text{ for } \nu\neq 0.$$

Here we never used the fact that $q=\alpha^{-2}$, so the summation of the inner series has nothing to do with Fibonacci numbers.

We hope to get some of the Fibonacci enthusiasts interested in theta functions. An excellent text is Rademacher's lecture notes [6]. They pair German thoroughness with elegance. On the other side of the spectrum is Bellman's very thin book [2], which contains almost no proofs, only the most important results and some applications.

1. THETA FUNCTIONS

We have the following theta functions (the summation is over all n in $\mathbb{Z}$):

$$\vartheta_1(x,\ q)=\frac{1}{i}\sum_n(-1)^n q^{[(n+(1/2)]^2}e^{i(2n+1)\pi x};$$

$$\vartheta_2(x,\ q)=\sum_n q^{[n+(1/2)]^2}e^{i(2n+1)\pi x};$$

$$\vartheta_3(x,\ q)=\sum_n q^{n^2}e^{i2n\pi x};$$

$$\vartheta_4(x,\ q)=\sum_n(-1)^n q^{n^2}e^{i2n\pi x}.\tag{1}$$

We make the substitution $q=e^{\pi i z}$ and get the following *functional equations*:

$$\vartheta_1\left(\frac{x}{z},\ -\frac{1}{z}\right)=\frac{1}{i}\sqrt{\frac{z}{i}}\,e^{(\pi i x^2/z)}\vartheta_1(x,\ z);$$

and

$$\vartheta_{3+\nu}\left(\frac{x}{z},\ -\frac{1}{z}\right)=\sqrt{\frac{z}{i}}\,e^{(\pi i x^2/z)}\vartheta_{3-\nu}(x,\ z)\text{ for }\nu=-1,\ 0,\ 1.\tag{2}$$

($\sqrt{i}$ is taken in the first quadrant.)

This was essentially proved in 1823 by Poisson in the form:

$$\frac{1}{\sqrt{x}}=\frac{1+2e^{-\pi x}+2e^{-4\pi x}+2e^{-9\pi x}+\cdots}{1+2e^{-\pi/x}+2e^{-4\pi/x}+2e^{-9\pi/x}+\cdots}.$$

Notation: In the sequel we will only have to consider the case $x=0$. We will write

$$\vartheta_\nu=\vartheta_\nu(0,\ q),$$

$$\vartheta_\nu^{(n)}=\left(\frac{\partial}{\partial x}\right)^n\vartheta(0,\ q).$$

We have many formulas, as follows:

$$\vartheta_1' = \pi \sum_n (-1)^n (2n + 1) q^{[n+(1/2)]^2};$$

$$\vartheta_2 = \sum_n q^{[n+(1/2)]^2};$$

$$\vartheta_3 = \sum_n q^{n^2};$$

$$\vartheta_4 = \sum_n (-1)^n q^{n^2}, \quad \vartheta_4(-q) = \vartheta_3(q). \tag{3}$$

$$\vartheta_1''' = -\pi^3 \sum_n (-1)^n (2n + 1)^3 q^{[n+(1/2)]^2};$$

$$\vartheta_2'' = -\pi^2 \sum_n (2n + 1)^2 q^{[n+(1/2)]^2};$$

$$\vartheta_3'' = -4\pi^2 \sum_n n^2 q^{n^2};$$

$$\vartheta_4'' = -4\pi^2 \sum_n (-1)^n n^2 q^{n^2}. \tag{4}$$

By using the transformed formulas, we get:

$$\vartheta_1' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n (-1)^n \left(n - \frac{1}{2}\right) e^{\frac{\pi^2}{\log q}[n-(1/2)]^2};$$

$$\vartheta_2 = \sqrt{-\frac{\pi}{\log q}} \sum_n (-1)^n e^{\frac{\pi^2}{\log q}(n^2)};$$

$$\vartheta_3 = \sqrt{-\frac{\pi}{\log q}} \sum_n e^{\frac{\pi^2 n^2}{\log q}};$$

$$\vartheta_4 = \sqrt{-\frac{\pi}{\log q}} \sum_n e^{\frac{\pi^2[n-(1/2)]^2}{\log q}};$$

$$\vartheta_2'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2 n^2}{\log q}\right) e^{\pi^2 n^2/\log q} (-1)^n;$$

$$\vartheta_3'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2 n^2}{\log q}\right) e^{\pi^2 n^2/\log q};$$

$$\vartheta_4'' = \frac{2\pi^2}{\log q} \sqrt{-\frac{\pi}{\log q}} \sum_n \left(1 + \frac{2\pi^2}{\log q}\left(n - \frac{1}{2}\right)^2\right) e^{\frac{\pi^2[n-(1/2)]^2}{\log q}}. \tag{5}$$

2. COMPUTATION OF THE SUM $\sum_0^\infty \frac{1}{L_{2n} + 2}$

We will go through the computation of the above sum in detail. We put, as usual,

$$\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}.$$

Then $\alpha\beta = -1$ and $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, $L_n = \alpha^n + \beta^n$. If we put $q = \alpha^{-1}$, we get:

$$S = \sum_0^\infty \frac{1}{L_{2n} + 2} = \frac{1}{4} + \sum_1^\infty \frac{1}{\alpha^{2n} + \alpha^{-2n} + 2} = \frac{1}{4} + \sum_1^\infty \frac{q^{2n}}{(1 + q^{2n})^2}.$$

By by formulas in Tannery and Molk [7, II, pp. 250 and 260] we have

$$\frac{\vartheta_2''}{\vartheta_2} = -\pi^2\left(1 + 8\sum_1^\infty \frac{q^{2n}}{(1 + q^{2n})^2}\right).$$

Hence,

$$S = \frac{1}{8}\left(1 - \frac{1}{\pi^2} \cdot \frac{\vartheta_2''}{\vartheta_2}\right),$$

and if we use formulas (3) and (4), we get:

$$S = \frac{1}{8}\left(1 + \frac{\sum_{n=1}^\infty (2n + 1)^2\alpha^{-[n+(1/2)]^2}}{\sum_1^\infty \alpha^{-[n+(1/2)]^2}}\right).$$

This series converges very rapidly (10 terms will give about 20 decimal places) but it does not contain $\log \alpha$ as Backstrom's approximation does. By using the functional equation and the formulas in (5) we can improve the rate of convergence.

$$S = \frac{1}{8}\left(1 - \frac{2}{\log q}\,\frac{\sum_n (-1)^n\left(1 + \frac{2\pi^2 n^2}{\log q}\right)e^{\pi^2 n^2/\log q}}{\sum_n (-1)^n e^{\pi^2 n^2/\log q}}\right).$$

Putting $q = \alpha^{-1}$, we obtain the final formula:

$$\sum_0^\infty \frac{1}{L_{2n} + 2} = \frac{1}{8} + \frac{1}{4\log \alpha}\left[1 - \frac{4\pi^2}{\log \alpha} \cdot \frac{\sum_1^\infty (-1)^n n^2 e^{-\pi^2 n^2/\log \alpha}}{1 + 2\sum_1^\infty (-1)^n e^{-\pi^2 n^2/\log \alpha}}\right].$$

This series converges extremely rapidly. We have $e^{-\pi^2/\log \alpha} \approx e^{-20} \approx 10^{-9}$, so taking just one term ($n = 1$) will give over 30 correct decimal places. Ten terms will give around 900 correct decimal places.

3. A CATALOGUE OF FORMULAS

In this section we collect some formulas connecting sums of reciprocals of Fibonacci and Lucas numbers and theta functions. We leave it to the reader to derive the final formulas as in the last section. The formulas are found in Tannery and Molk [7, II, pp. 250, 260, 258; IV, pp. 108, 107], Jacobi [4, pp. 159-167], and Hancock [3, p. 407].

I. Put $q = \alpha^{-1}$. Then:

(a) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n} + 2} = \frac{1}{8}(\vartheta_3^2\vartheta_4^2 - 1)$

(b) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n}} = \frac{1}{4}(\vartheta_3\vartheta_4 - 1)$

(c) $\displaystyle\sum_1^\infty \frac{n^3}{F_{2n}} = \frac{\sqrt{5}}{256}\vartheta_2^8$

(d) $\displaystyle\sum_1^\infty \frac{2n-1}{L_{2n-1}} = \frac{1}{16}\vartheta_2^4$

II. Put $q = \alpha^{-2}$. Then:

(a) $\displaystyle\sum_1^\infty \frac{1}{F_{2n-1}} = \frac{\sqrt{5}}{4}\vartheta_2^2$

(b) $\displaystyle\sum_1^\infty \frac{1}{L_{2n}} = \frac{1}{4}(\vartheta_3^2 - 1)$

(c) $\displaystyle\sum_1^\infty \frac{(-1)^{n-1}(2n-1)}{F_{2n-1}} = \frac{\sqrt{5}}{4}\vartheta_2^2\vartheta_4^2$

(d) $\displaystyle\sum_1^\infty \frac{2n-1}{L_{2n-1}} = \frac{1}{4}\vartheta_2^2\vartheta_4^2$

(e) $\displaystyle\sum_1^\infty \frac{(-1)^n}{L_{2n}^2} = \frac{1}{8}(\vartheta_3^2\vartheta_4^2 - 1)$

(f) $\displaystyle\sum_1^\infty \frac{1}{F_{2n}^2} = \frac{5}{24}\left(1 + \frac{1}{\pi^2}\frac{\vartheta_1''}{\vartheta_1'}\right)$

(g) $\displaystyle\sum_1^\infty \frac{1}{L_{2n}^2} = -\frac{1}{8}\left(1 + \frac{1}{\pi^2}\frac{\vartheta_2''}{\vartheta_2}\right)$

(h) $\displaystyle\sum_1^\infty \frac{1}{F_{2n-1}^2} = -\frac{5}{8\pi^2}\frac{\vartheta_3''}{\vartheta_3}$

(i) $\displaystyle\sum_1^\infty \frac{1}{L_{2n-1}^2} = \frac{1}{8\pi^2}\frac{\vartheta_4''}{\vartheta_4}$

(j) $\displaystyle\sum_1^\infty \frac{n}{F_{2n}} = \frac{\sqrt{5}}{8\pi^2}\frac{\vartheta_4''}{\vartheta_4}$

4. SOME IDENTITIES

There are numerous identities among theta functions. Specializing to $q = \alpha^{-1}$ or $q = \alpha^{-2}$ will give identities among sums of Fibonacci and Lucas numbers. We will give a few examples.

(a) Formulas II(i) and (j) give two expressions for $\vartheta_4''/\vartheta_4$:

$$\sum_1^\infty \frac{n}{F_{2n}} = \sqrt{5}\sum_1^\infty \frac{1}{L_{2n-1}^2}$$

(b) Formulas I(d) and II(a) give, with $q = \alpha^{-2}$,

$$\left(\sum_1^\infty \frac{1}{F_{2n-1}}\right)^2 = 5\sum_1^\infty \frac{2n-1}{L_{4n-2}}$$

(c) The identity (Tannery and Molk [7, II, p. 250])

$$3\sum_1^\infty \frac{q^{2n}}{(1-q^{2n})^2} = \sum_1^\infty \frac{q^{2n-1}}{(1-q^{2n-1})^2} - \sum_1^\infty \frac{q^{2n-1}}{(1+q^{2n-1})^2} - \sum_1^\infty \frac{q^{2n}}{(1+q^{2n})^2}$$

gives, with $q = \alpha^{-2}$:

$$3\sum_1^\infty \frac{1}{F_{2n}^2} + \sum_1^\infty \frac{1}{F_{2n-1}^2} = 5\left(\sum_1^\infty \frac{1}{L_{2n-1}^2} - \sum_1^\infty \frac{1}{L_{2n}^2}\right)$$

(d) We have $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$, which implies:

$$\left(1 + 4\sum_1^\infty \frac{1}{L_{2n}}\right)^2 = \frac{16}{5}\left(\sum_1^\infty \frac{1}{F_{2n-1}}\right)^2 + \left(1 + 4\sum_1^\infty \frac{(-1)^n}{L_{2n}}\right)^2$$

5. A NEW TANTALIZING QUESTION

Unfortunately, we have not been able to find an expression for the sum

$$\sum_1^\infty \frac{1}{F_n}.$$

Since we know from II(a) that

$$\sum_1^\infty \frac{1}{F_{2n-1}} = \frac{\pi\sqrt{5}}{8\log\alpha}\left\{1 + 2\sum_1^\infty (-1)^n e^{-\pi^2 n^2/2\log\alpha}\right\}^2$$

we only need to compute $\sum_1^\infty \frac{1}{F_{2n}}$. For this, we need (wtih $q = \alpha^{-2}$)

$$f(q) = \sum_1^\infty \frac{q^n}{1 - q^{2n}} = \sum_1^\infty \frac{q^{2n-1}}{1 - q^{2n-1}} = \sum_1^\infty T_0(n)q^n,$$

where $T_0(n)$ is the number of odd divisors of n. Since T_0 is multiplicative, i.e., $T_0(mn) = T_0(m)T_0(n)$ if $(m, n) = 1$, we can compute the Dirichlet series (for Re $s > 1$).

$$\Phi(s) = \sum_{n=1}^\infty \frac{T_0(n)}{n^s} = \prod_p\left(\sum_{\nu\geqslant 0} \frac{T_0(p^\nu)}{p^{\nu s}}\right),$$

where the product is taken over all prime numbers. We have

$$T_0(2^\nu) = 1 \quad\text{and}\quad T_0(p^\nu) = \nu + 1 \quad\text{if } p \geqslant 3.$$

Hence, putting $t = p^{-s}$, we have

$$\sum_0^\infty T_0(2^\nu)t^\nu = \frac{1}{1-t}$$

and

$$\sum_0^\infty T_0(p^\nu)t^\nu = \sum_0^\infty (\nu + 1)t^\nu = \frac{1}{(1-t)^2} \quad\text{for } p \geqslant 3.$$

It follows that

$$\Phi(s) = \frac{1}{1 - 2^{-s}}\prod_{p\geqslant 3}\frac{1}{(1 - p^{-s})^2} = (1 - 2^{-s})\zeta(s)^2,$$

where

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s}$$

is the Riemann ζ-function.

It is possible, at least theoretically, to recover f from Φ by Mellin inversion (see Ogg [5, p. I.6]); however, we have not been able to compute the integral.

We end by giving some formulas due to Clausen (see Jacobi [4, I, p. 239]): Put

$$h(q) = \sum_1^\infty \frac{q^n}{1 - q^n} = \sum_1^\infty q^{n^2}\frac{1 + q^n}{1 - q^n}$$

What we need is

$$\sum_1^\infty \frac{q^{2n-1}}{1 - q^{2n-1}} = h(q) - h(q^2) = \sum_1^\infty \left(q^{n^2}\frac{1 + q^n}{1 - q^n} - q^{2n^2}\frac{1 + q^{2n}}{1 - q^{2n}} \right)$$

which converges very rapidly when $q = \alpha^{-2}$.

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