A NOTE CONCERNING THE NUMBER OF ODD-ORDER MAGIC SQUARES

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INTRODUCTION

A square array of consecutive integers $1, 2, \ldots, n^2$ is called *magic of order* n if the rows, columns, and diagonals all add up to the same number. If in addition, the sum of the numbers in each broken diagonal is also the same number, then the magic square is said to be *pandiagonal*. Let M be a magic square of order n and let its entries by denoted by (i, j), $(1 \leqslant i, j \leqslant n)$. Then M is *symmetrical* if $(i, j) + (n - i + 1, n - j + 1) = n^2 + 1$. Let D_4 denote the dihedral group of order 8. Then, two magic squares M and M' are said to be *equivalent* if there is a σ in D_4 such that $\sigma(M) = M'$.

Let

$\sigma_0(n)$ = number of inequivalent magic squares of order n.

$\delta_0(n)$ = number of inequivalent pandiagonal magic squares of order n.

$\rho_0(n)$ = number of inequivalent symmetrical magic squares of order n.

$\gamma_0(n)$ = number of inequivalent pandiagonal and symmetrical magic squares of order n.

While it is not difficult to construct, for any $n \geqslant 3$, a magic square of order n, it seems formidable to determine $\sigma_0(n)$ or $\delta_0(n)$ for $n \geqslant 6$ (see [1] and [2]). In [4], it is shown that $\delta_0(4) = 48$ and in [5] that $\delta_0(5) = 3600$. In this note, we shall show that, given an odd-order pandiagonal magic square, we can use it to generate a finite iterative sequence of pandiagonal magic squares of the same order. We show that the number of terms in this sequence is always even. It is observed that, if we start with a non-pandiagonal magic square of odd order, then magic squares and non-magic squares occur alternatively in the sequence. It is also observed that if the initial square is symmetrical, then so is the next one. We then determine the number of terms in the above iterative sequences, thereby showing that each of $\sigma_0(n)$, $\rho_0(n)$, $\delta_0(n)$, and $\gamma_0(n)$ is a multiple of the number of terms in its respective sequence. Finally, we note that our results may be combined with others to yield stronger results.

RESULTS

Let M be a pandiagonal magic square of order n. Obtain from M a square $\varphi(M)$ whose entries $\varphi(i, j)$, $(1 \leqslant i, j \leqslant n)$, are given by

$$\varphi(i, j) = (m + 1 + i - j, m + i + j),$$

where $m = (n - 1)/2$ and the operations are taken modulo n. Then it is routine to verify that $\varphi(M)$ is magic and pandiagonal (see [3]). Further, if M is symmetrical, then so is $\varphi(M)$. For $r > 1$, define, inductively,

$$\varphi^r(M) = \varphi(\varphi^{r-1}(M)).$$

Thus, we obtain a sequence M, $\varphi(M)$, ..., of pandiagonal magic squares of order n. Note that φ is one-to-one and onto and hence its inverse exists. Lemma 1, below, asserts that the sequences generated by M and σM under φ are equivalent. Further, there exists r such that $\sigma\varphi^r(M) = M$ for some $\sigma \in D_4$. We wish to determine the smallest such r.

Lemma 1: Let $\sigma \in D_4$. Then $\varphi(\sigma(M)) = \pi\varphi(M)$ for some $\pi \in D_4$.

Proof: If σ is a 90° clockwise rotation, then $\sigma(i, j) = (j, n - i + 1)$. It is routine to verify that $\varphi\sigma^k(i, j) = \sigma^k\varphi(i, j)$, where $k = 0, 1, 2, 3$. If σ is the reflection along the central vertical (horizontal), then $\sigma(i, j) = (i, n - j + 1)$ [respectively, $(n - i + 1, j)$]. Choose π to be the diagonal reflection with $\pi(i, j) = (j, i)$ [respectively, $(n - j + 1, n - i + 1)$]. If σ is a diagonal reflection, then $\sigma(i, j) = (j, i)$ or $\sigma(i, j) = (n - j + 1, n - i + 1)$, in which case let π be the reflection along the central horizontal and central vertical, respectively. This completes the proof.

Let the entries of $\varphi^r(M)$ by denoted by $\varphi^r(i, j)$. Then it is easy to verify that $\varphi^r(i, j)$ is given by the following:

If $r = 2s$, $s \geqslant 1$, then

$$\varphi^r(i, j) = \begin{cases} (m + 1 + 2^{s-1} - 2^s j,\ m + 1 - 2^{s-1} + 2^s i) & s \equiv 1 \pmod 4, \\ (m + 1 + 2^{s-1} - 2^s i,\ m + 1 + 2^{s-1} - 2^s i) & s \equiv 2 \pmod 4, \\ (m + 1 - 2^{s-1} + 2^s j,\ m + 1 + 2^{s-1} - 2^s i) & s \equiv 3 \pmod 4, \\ (m + 1 - 2^{s-1} + 2^s i,\ m + 1 - 2^{s-1} + 2^s j) & s \equiv 0 \pmod 4. \end{cases}$$

If $r = 2s + 1$, $s \geqslant 0$, then

$$\varphi^r(i, j) = \begin{cases} (m + 1 + 2^s - 2^s(i + j),\ m + 1 + 2^s(i - j)) & s \equiv 1 \pmod 4, \\ (m + 1 - 2^s(i - j),\ m + 1 + 2^s - 2^s(i + j)) & s \equiv 2 \pmod 4, \\ (m + 1 - 2^s + 2^s(i + j),\ m + 1 - 2^s(i - j)) & s \equiv 3 \pmod 4, \\ (m + 1 + 2^s(i - j),\ m + 1 - 2^s + 2^s(i + j)) & s \equiv 0 \pmod 4. \end{cases}$$

The proof of the following lemma is straightforward and so is omitted.

Lemma 2: Suppose n is odd and $n = 2m + 1$.

(i) If $2^s \equiv 1 \pmod n$, then $m + 1 - 2^{s-1} \equiv 0 \pmod n$ and $m + 1 + 2^{s-1} \equiv 1 \pmod n$.

(ii) If $2^s \equiv -1 \pmod n$, then $m + 1 - 2^{s-1} \equiv 1 \pmod n$ and $m + 1 + 2^{s-1} \equiv 0 \pmod n$.

Proposition 1: Let n be odd. Then $\delta_0(n) \equiv 0 \pmod 2$ and $\gamma_0(n) \equiv 0 \pmod 2$.

Proof: If r is odd, then $(1, 1)$ will be an entry in the central column or central row of $\varphi^r(M)$. This means that there is no σ in D_4 such that $\sigma\varphi^r(1, 1) = (1, 1)$. The result thus follows.

Proposition 2: Let n be an odd number. Suppose s is the smallest integer such that $2^s \equiv 1 \pmod{n}$ or $2^s \equiv -1 \pmod{n}$. Then

(i) $\sigma_0(n) \equiv 0 \pmod{s}$,

(ii) $\rho_0(n) \equiv 0 \pmod{s}$,

(iii) $\delta_0(n) \equiv 0 \pmod{2s}$,

and (iv) $\gamma_0(n) \equiv 0 \pmod{2s}$.

Proof: We shall prove (iii). The proof of (iv) then follows immediately; that of (i) follows from Proposition 1 and the fact that if M is magic but not pandiagonal then $\varphi(M)$ is not magic but $\varphi^2(M)$ is magic; (ii) follows from the fact that $\varphi(M)$ is symmetrical if M is.

Let $r = 2k$.

Now $\varphi^r(1, 1)$ is one of:

$$(m + 1 - 2^{k-1}, m + 1 + 2^{k-1}),\ (m + 1 - 2^{k-1}, m + 1 - 2^{k-1}),$$
$$(m + 1 + 2^{k-1}, m + 1 - 2^{k-1}),\ (m + 1 + 2^{k-1}, m + 1 + 2^{k-1}).$$

If $r < 2s$, then we see that $m + 1 + 2^{k-1}$ cannot be $0 \pmod{n}$ or $1 \pmod{n}$. Likewsie, $m + 1 - 2^{k-1}$ cannot be $0 \pmod{n}$ or $1 \pmod{n}$. So there is no σ in D_4 such that $\sigma\varphi^r(1, 1) = (1, 1)$.

Suppose $r = 2s$.

Now if $2^s \equiv 1 \pmod{n}$, then, by Lemma 2,

$$m + 1 - 2^{s-1} \equiv 0 \pmod{n} \quad \text{and} \quad m + 1 + 2^{s-1} \equiv 1 \pmod{n}.$$

If $2^s \equiv -1 \pmod{n}$, then

$$m + 1 - 2^{s-1} \equiv 1 \pmod{n} \quad \text{and} \quad m + 1 + 2^{s-1} \equiv 0 \pmod{n}.$$

In either case, $\varphi^r(i, j)$ is one of

$$(j, n - i + 1),\ (i, j),\ (n - j + 1, i),\ (n - i + 1, n - j + 1).$$

Certainly, there is a σ in D_4 such that $\sigma\varphi^r(i, j) = (i, j)$ and the result follows.

REMARKS

Note that there are other operations which will also generate finite sequences of inequivalent magic squares of the same order. For example:

(A) Cyclic permutation of the rows and/or columns of a pandiagonal magic square will produce an inequivalent pandiagonal magic square. Hence $\delta_0(n) \equiv 0 \pmod{n^2}$.

(B) Let $n = 2m + 1$. Then any permutation of the numbers $1, 2, \ldots, m$ applied to the first m rows and columns and to the last m rows and columns (in reverse order) of a magic square of order n will result in an inequivalent magic square. Further, if we start with a symmetrical square, then so are all other squares generated in this manner. Hence $\sigma_0(n) \equiv 0 \pmod{m!}$ and $\rho_0(n) \equiv 0 \pmod{m!}$.

More magic squares may be obtained by combining the operation φ with that of (A) or (B).

Proposition 3: Let $n = 2m + 1$ and suppose s satisfies the conditions of Proposition 2. Then

(i) $\sigma_0(n) \equiv 0 \pmod{s \cdot m!}$,

(ii) $\rho_0(n) \equiv 0 \pmod{s \cdot m!}$

and (iii) $\delta_0(n) \equiv 0 \pmod{2sn^2}$.

Proof: Let M be a pandiagonal magic square of order n. For each $\varphi^r(M)$, we apply the operation in (A) to get n^2 inequivalent pandiagonal magic squares. It remains to show that these n^2 squares are not equivalent to any of those generated by φ. To see this, it suffices to note that $(m + 1, m + 1)$ is always fixed under φ, while in the operation (A) it is being transferred to other positions. This proves (iii).

To prove (i) and (ii), let M be a magic square of order n. For each $\varphi^{2k}(M)$ (which is magic), we apply the operation in (B) to get $m!$ inequivalent magic squares. We shall show that these $m!$ squares are not equivalent to any one of those generated by φ. Since the operation φ transfers the central row and the central column of M to the main diagonals of the resulting square, it follows that we need only consider $\varphi^{2k}(M)$. Consider the entries $(i, m + 1)$, where $i = 1, 2, \ldots, m$. If k is odd, then $\varphi^{2k}(i, m + 1) = (m + 1, x + 2yi)$ for some integers x and y. If k is even, then $\varphi^{2k}(i, m + 1) = (x - 2yi, m + 1)$. However, under the operation in (B), the entries $(i, m + 1)$, where $i = 1, 2, \ldots, m$ go to $(\sigma(i), m + 1)$ for some permutation σ of the numbers $1, 2, \ldots, m$. This means that $\varphi^{2k}(M)$ cannot be equivalent to any one of the squares generated by the operation in (B).

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