ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications concerning ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN; 709 SOLANO DR., S.E.; ALBUQUERQUE, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date. Proposed problems should be accompanied by their solutions.

DEFINITIONS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1$$

and

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

PROBLEMS PROPOSED IN THIS ISSUE

B-580 *Proposed by Valentina Bakinova, Rondout Valley, NY*

What are the three smallest positive integers d such that no Lucas number L_n is an integral multiple of d?

B-581 *Proposed by Antal Bege, University of Cluj, Romania*

Prove that, for every positive integer n, there are at least $[n/2]$ ordered 6-tuples (a, b, c, x, y, z) such that

$$F_n = ax^2 + by^2 - cz^2$$

and each of a, b, c, x, y, z is a Fibonacci number. Here $[t]$ is the greatest integer in t.

B-582 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

It is known that every positive integer N can be represented uniquely as a sum of distinct nonconsecutive positive Fibonacci numbers. Let $f(N)$ be the number of Fibonacci addends in this representation, $\alpha = (1 + \sqrt{5})/2$, and $[x]$ be the greatest integer in x. Prove that

$$f([\alpha F_n^2]) = [(n + 1)/2] \text{ for } n = 1, 2, \ldots .$$

B-583 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

For positive integers n and s, let

$$S_{n,s} = \sum_{k=1}^{n} \binom{n}{k} k^s.$$

Prove that $S_{n,s+1} = n(S_{n,s} - S_{n-1,s})$.

B-584 *Proposed by Dorin Andrica, University of Cluj-Napoca, Romania*

Using the notation of B-583, prove that

$$S_{m+n,s} = \sum_{k=0}^{s} \binom{s}{k} S_{m,k} S_{n,s-k}.$$

B-585 *Proposed by Constantin Gonciulea & Nicolae Gonciulea, Trian College, Drobeta Turnu-Severin, Romania*

For each subset A of $X = \{1, 2, \ldots, n\}$, let $r(A)$ be the number of j such that $\{j, j+1\} \subseteq A$. Show that

$$\sum_{A \subseteq X} 2^{r(A)} = F_{2n+1}.$$

SOLUTIONS

Pattern for Squares

B-556 *Proposed by Valentina Bakinova, Rondout Valley, NY*

State and prove the general result illustrated by

$$4^2 = 16,\ 34^2 = 1156,\ 334^2 = 111556,\ 3334^2 = 11115556.$$

Solution by Thomas M. Green, Contra Costa College, San Pablo, CA

Let $D_n = 1 + 10 + 10^2 + \cdots + 10^{n-1}$. The general result

$$(3D_n + 1)^2 = 10^n D_n + 5D_n + 1$$

is proved by expanding the left member and observing that $9D_n = 10^n - 1$.

Note: The quantity D_n has several other interesting properties:

(i) $D_n = 111\ldots111$ (n ones)

(ii) $D_n^2 = 123\ldots n\ldots321$ ($n = 1, \ldots, 9$)

(iii) $D_9/9 = 123456789$

(iv) $(b-1)D_n = b^n - 1$ (b is your number base)

(v) The sequence $D_2^0, D_2^1, D_2^2, \ldots, D_2^n$, is Pascal's triangle (with suitable restrictions on carrying) and the sequences $D_n^0, D_n^1, D_n^2, \ldots, D_n^r$, are Pascal-like triangles where each entry is the sum of the n entries above it.

Also solved by Paul S. Bruckman, Lászlo Cseh, L. A. G. Dresel, Piero Filipponi, J. Foster, Herta T. Freitag, Hans Kappus, H. Klauser, L. Kuipers, Graham Lord, *Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.*

Not True Any Year

B-557 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there is no integer $n \geqslant 2$ such that

$$F_{3n-6}F_{3n-3}F_{3n+3}F_{3n+6} - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1985^8 + 1.$$

Solution by J. Suck, Essen, Germany

Since $F_{3k} \equiv 2F_k \pmod 3$ [see, e.g., B-182, *The Fibonacci Quarterly* 8 (Dec. 1970), for the more general $F_{pk} \equiv F_pF_k \pmod p$, p a prime], the left-hand side is congruent to $(2^4 - 1)F_{n-2}F_{n-1}F_{n+1}F_{n+2}$, hence to 0. But the right-hand side is not whatever the year may be: if $y \equiv 0, 1, 2$, then $y^8 + 1 \equiv 1, 2, 2$, respectively, mod 3.

Also solved by Paul S. Bruckman, Lászlo Cseh, L. A. G. Dresel, Piero Filipponi, J. Foster, L. Kuipers, Sahib Singh, M. Wachtel, and the proposer.

Impossible Equation

B-558 *Proposed by Imre Merényi, Cluj, Romania*

Prove that there are no positive integers m and n such that

$$F_{4m}^2 - F_{3n} - 4 = 0.$$

Solution by L. A. G. Dresel, University of Reading, England

Since $F_3 = 2$ and $F_6 = 8$, we have $F_{3n} \equiv 2 \pmod 4$ when n is odd, and $F_{3n} \equiv 0$ when n is even. Now consider the equation $F_{4m}^2 = F_{3n} + 4$. Clearly n cannot be odd, since $F^2 \equiv 2 \pmod 4$ is not possible. However, if n is even, $F_{4m}^2 \equiv 4 \pmod 8$ and this implies $F_{4m} \equiv \pm 2 \pmod 8$. Hence F_{4m} is even, so that $m = 3k$, where k is an integer, and therefore $F_{4m} = F_{12k}$, which is divisible by 8. This contradicts $F_{4m} = \pm 2 \pmod 8$. Hence there are no integers m and n such that $F_{4m}^2 - F_{3n} - 4 = 0$.

We note that the above argument actually proves the slightly stronger result that there are no integers m and n such that $F_{2m}^2 - F_{3n} - 4 = 0$.

Also solved by Paul S. Bruckman, Lászlo Cseh, Piero Filipponi, L. Kuipers, Sahib Singh, Lawrence Somer, M. Wachtel, and the proposer.

Golden Mean Identity

B-559 *Proposed by Lászlo Cseh, Cluj, Romania*

Let $\alpha = (1 + \sqrt{5})/2$. For positive integers n, prove that

$$[\alpha + .5] + [\alpha^2 + .5] + \cdots + [\alpha^n + .5] = L_{n+2} - 2,$$

where $[x]$ denotes the greatest integer in x.

Solution by J. Foster, Weber State College, Ogden, UT

Since $L_k = \alpha^k + \beta^k$ and, for $k \geqslant 2$, $[.5 - \beta^k] = 0$,

$$\sum_{k=1}^{n} [\alpha^k + .5] = \sum_{k=1}^{n} [L_k - \beta^k + .5] = \sum_{k=1}^{n} (L_k + [.5 - \beta^k])$$

$$= \sum_{k=0}^{n} L_k - L_0 + [.5 - \beta] = (L_{n+2} - 1) - 2 + 1 = L_{n+2} - 2.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, C. Georghiou, Hans Kappus, L. Kuipers, Graham Lord, Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Another Greatest Integer Identity

B-560 *Proposed by Lászlő Cseh, Cluj, Romania*

Let α and $[x]$ be as in B-559. Prove that

$$[\alpha F_1 + .5] + [\alpha^2 F_2 + .5] + \cdots + [\alpha^n F_n + .5]$$

is always a Fibonacci number.

Solution by C. Georghiou, University of Patras, Greece

We have

$$\alpha^n F_n = \frac{\alpha^{2n} - (-1)^n}{\sqrt{5}} = F_{2n} + \frac{\beta^{2n} - (-1)^n}{\sqrt{5}}$$

and since

$$\left[\frac{\beta^{2n} - (-1)^n}{\sqrt{5}} + .5\right] = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

the given sum becomes

$$F_2 + 1 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1}.$$

Also solved by Paul S. Bruckman, L. A. G. Dresel, Piero Filipponi, J. Foster, Hans Kappus, L. Kuipers, Imre Merényi, Bob Prielipp, Sahib Singh, Lawrence Somer, J. Suck, and the proposer.

Q-Matrix Identity

B-561 *Proposed by Piero Filipponi, Fond. U. Bordoni, Roma, Italy*

(i) Let Q be the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. For all integers n, show that

$$Q^n + (-1)^n Q^{-n} = L_n I, \text{ where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) Find a square root of Q, i.e., a matrix A with $A^2 = Q$.

Solution by Sahib Singh, Clarion University, Clarion, PA

(i) If $n = 0$, then $Q^0 + (Q^0)^{-1} = 2I = L_0 I$.

For $n \geqslant 1$, it follows by mathematical induction that:

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

$$Q^{-n} = (-1)^n \begin{bmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{bmatrix}$$

Thus, $Q^n + (-1)^n Q^{-n} = \begin{bmatrix} L_n & 0 \\ 0 & L_n \end{bmatrix} = L_n I$ for all $n \geqslant 1$.

Changing n to $-n$, the above equation becomes:

$$Q^{-n} + (-1)^{-n} Q^n = L_{-n} I \quad \text{or} \quad (-1)^n [Q^n + (-1)^n Q^{-n}] = (-1)^n L_n I,$$

so that

$$Q^n + (-1)^n Q^{-n} = L_n I \text{ for } n \leqslant -1.$$

Thus, the result holds for all integers.

(ii) Let a square root of Q be denoted by S where $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $S^2 = Q$ yields

$$a^2 + bc = 1; \ (a + d)b = 1; \ (a + d)c = 1; \ bc + d^2 = 0.$$

Solving these equations, we conclude that

$$a = \frac{1 + b^2}{2b}; \ c = b; \ d = \frac{1 - b^2}{2b}, \text{ where } b \text{ satisfies}$$

$$5b^4 - 2b^2 + 1 = 0.$$

Thus, a square root of Q is $\begin{bmatrix} \frac{1 + b^2}{2b} & b \\ b & \frac{1 - b^2}{2b} \end{bmatrix}$,

where b is a complex number satisfying $5b^4 - 2b^2 + 1 = 0$, which can be solved by the quadratic formula using $x = b^2$.

Also solved by Paul S. Bruckman, László Cseh, L. A. G. Dresel, J. Foster, C. Georghiou, Hans Kappus, L. Kuipers, Bob Prielipp, Lawrence Somer, J. Suck, and the proposer.

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