A NOTE ON THE REPRESENTATION OF INTEGERS AS A SUM OF DISTINCT FIBONACCI NUMBERS

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1. INTRODUCTION AND GENERALITIES

It is known that every positive integer can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). A series of papers published over the past years deal with this subject and related problems [1, 2, 3, 4]. Our purpose in this note is to investigate some minor aspects of this property of the Fibonacci sequence. More precisely, for a given integer $k \geqslant 3$, we consider the set $\mathcal{N}_k$ of all positive integers n less than F_k (as usual F_k and L_k are the k^{th} Fibonacci and Lucas numbers, respectively), and for these integers we determine:

(i) the asymptotic value of the average number of F-addends;

(ii) the most probable number of F-addends;

(iii) the greatest number m_k of F-addends, selected from the set $\mathcal{N}_k$, and the integers representable as a sum of m_k F-addends.

Setting

$$m_k = [(k - 1)/2], \quad (k \geqslant 3) \tag{1}$$

(here and in the following the symbol $[x]$ denotes the greatest integer not exceeding x) and denoting by $f(n, k)$ the number of F-addends the sum of which represents a generic integer $n \in \mathcal{N}_k$, we state the following theorems.

Theorem 1: $1 \leqslant f(n, k) \leqslant m_k$.

Proof: Since $F_1 = F_2$ and since the F-addends are distinct, they can be chosen in the set $\mathcal{F}_k = \{F_2, F_3, \ldots, F_{k-1}\}$ the cardinality of which is $|\mathcal{F}_k| = k - 2$. Moreover, since the F-addends are nonconsecutive Fibonacci numbers, they can be in number at most either $|\mathcal{F}_k|/2$ (for $|\mathcal{F}_k|$ even) or $(|\mathcal{F}_k| + 1)/2$ (for $|\mathcal{F}_k|$ odd). Q.E.D.

Theorem 2: The number $N_{k,m}$ of integers belonging to $\mathcal{N}_k$ which can be represented as a sum of m F-addends is given by

$$N_{k,m} = \binom{k - m - 1}{m}.$$

Proof: Setting $M = |\mathcal{F}_k| = k - 2$, it is evident that $N_{k,m}$ equals the number $B_{M,m}$ of distinct binary sequences of length M containing m nonadjacent 1's and $M - m$ 0's. The number $B_{M,m}$ can be obtained by considering the string

$$\{v\ 0\ v\ 0\ v \cdots v\ 0\ v\}$$

constituted by $M - m$ 0's and $M - m + 1$ empty elements v, and by replacing, in all possible ways, m empty elements by m 1's:

$$B_{M,m} = \binom{M - m + 1}{m}.$$

Replacing M by $k - 2$ in the above relation, the theorem is proved. Q.E.D.

From Theorem 2, we derive immediately the following

Remark:

$$N_{k,m} = \begin{cases} k - 2, & \text{for } m = 1 \\ 0, & \text{for } m > m_k. \end{cases} \tag{2}$$

2. THE AVERAGE VALUE OF $f(n, k)$

In this section, we calculate the limit of the ratio between the average value of $f(n, k)$ and k as k tends to infinity.

From Theorem 2, it is immediately seen that the average value $\overline{f}(n, k)$ of the number of F-addends the sum of which represents the integers belonging to $\mathcal{N}_k$ is given by

$$\overline{f}(n, k) = \frac{1}{|\mathcal{N}_k|}\sum_{m=1}^{m_k} mN_{k,m} = \frac{1}{F_k - 1}\sum_{m=1}^{\left[\frac{k-1}{2}\right]} m\binom{k - m - 1}{m}. \tag{3}$$

Moreover, it is known [5] that the identity

$$\sum_{m=0}^{m_k} (k - m)N_{k,m} = U_k \tag{4}$$

holds, where

$$U_k = \sum_{m=0}^{k-1} F_{m+1}F_{k-m}; \tag{5}$$

from (4), the relation

$$U_k = k\sum_{m=0}^{m_k} N_{k,m} - \sum_{m=0}^{m_k} mN_{k,m}$$

is obtained from which, by virtue of the well-known representation of the Fibonacci numbers as sums of binomial coefficients [6], we get

$$U_k = kF_k - \sum_{m=0}^{m_k} mN_{k,m}.$$

Consequently, we can write

$$\sum_{m=0}^{m_k} mN_{k,m} = \sum_{m=1}^{m_k} mN_{k,m} = kF_k - U_k. \tag{6}$$

The numbers U_k defined by (5) satisfy the recurrence stated in the following theorem.

Theorem 3: $U_k = kF_k - U_{k-2}$, with $U_1 = 1$, $U_2 = 2$.

Proof: Using the well-known identity $F_{s+t} = F_{s+1}F_t + F_sF_{t-1}$ and setting $m = s$, $k - m = t$, we can write the identity

$$F_k = F_{m+k-m} = F_{m+1}F_{k-m} + F_mF_{k-m-1}$$

thus getting $F_{m+1}F_{k-m} = F_k - F_mF_{k-m-1}$. Therefore, from (5), we have

$$U_k = \sum_{m=0}^{k-1}(F_k - F_mF_{k-m-1}) = kF_k - \sum_{m=0}^{k-1} F_mF_{k-m-1} = kF_k - \sum_{m=1}^{k-2} F_mF_{k-m-1}.$$

Setting $r = m - 1$, from the previous relation we obtain

$$U_k = kF_k - \sum_{r=0}^{k-3} F_{r+1}F_{k-r-2} = kF_k - U_{k-2}. \quad \text{Q.E.D.}$$

From Theorem 3, the further expression of U_k is immediately derived:

$$\begin{aligned} U_k &= kF_k - (k-2)F_{k-2} + \cdots + (-1)^{m_k}(k - 2m_k)F_{k-2m_k} \\ &= \sum_{i=0}^{m_k} (-1)^i (k - 2i)F_{k-2i}, \end{aligned} \tag{7}$$

where, as usual, $m_k = [(k-1)/2]$.

Denoting by α and β the roots of the equation $x^2 - x - 1 = 0$, the following theorem can be stated.

Theorem 4: $\bar{f}(n, k)$ is asymptotic to $\dfrac{1}{1 + \alpha^2}$.

Proof: From (3) and (6), we can write

$$\bar{f}(n, k)/k = \left(\frac{1}{F_k - 1}(kF_k - U_k)\right)\Big/k$$

and calculate the limit

$$\lim_{k\to\infty} \bar{f}(n, k)/k = \lim_{k\to\infty}\left(k - \frac{U_k}{F_k}\right)\Big/k = \lim_{k\to\infty} 1 - \frac{U_k}{kF_k}$$

which, from (7), can be rewritten as

$$\lim_{k\to\infty} \bar{f}(n, k)/k = \lim_{k\to\infty}\left(kF_k - kF_k + \sum_{i=1}^{m_k} (-1)^{i-1}(k - 2i)F_{k-2i}\right)\Big/(kF_k).$$

Finally, using the Binet form for F_k, we get

$$\begin{aligned} \lim_{k\to\infty} \bar{f}(n, k)/k &= \lim_{k\to\infty} \frac{\sum_{i=1}^{m_k}(-1)^{i-1}(k - 2i)(\alpha^{k-2i} - \beta^{k-2i})}{k(\alpha^k - \beta^k)} \\ &= \lim_{k\to\infty} \frac{\sum_{i=1}^{m_k}(-1)^{i-1}(1 - 2i/k)\alpha^{k-2i}}{\alpha^k} = \sum_{i=1}^{\infty} (-1)^{i-1}\alpha^{-2i} = \frac{1}{1 + \alpha^2} \approx 0.2764. \end{aligned}$$

Q.E.D.

The behavior of $\overline{f}(n, k)/k$ versus k has been obtained using a computer calculation and is shown in Figure 1 for $3 \leqslant k \leqslant 100$.

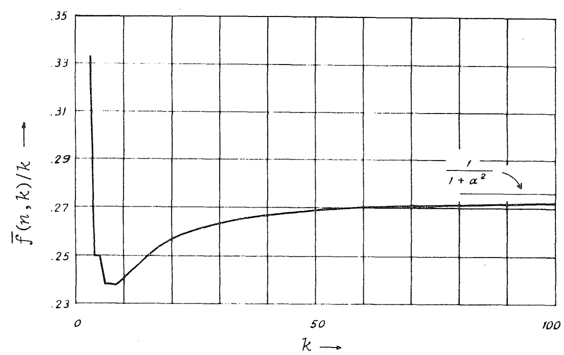

Figure 1. Behavior of $\overline{f}(n, k)/k$ versus k

3. THE MOST PROBABLE VALUE OF $f(n, k)$

In this section, it is shown that the most probable number $\hat{f}(n, k)$ of F-addends the sum of which represents the integers belonging to $\mathcal{N}_k$, can assume at most two (consecutive) values. The value of $\hat{f}(n, k)$ for a given k together with the values of k for which two $\hat{f}(n, k)$'s occur, are worked out.

From Theorem 2, it is immediately seen that $\hat{f}(n, k)$ equals the value(s) of m which maximize the binomial coefficient $N_{k,m}$; consequently let us investigate the behavior of the discrete function

$$\binom{h - n}{n} \tag{8}$$

as n varies, looking for the value(s) $\hat{n}_h$ of n which maximize it. It is evident that $\hat{n}_h$ is the value(s) of n for which the inequalities

$$\binom{h - n}{n} \geqslant \binom{h - n + 1}{n - 1} \tag{9}$$

and

$$\binom{h - n}{n} \geqslant \binom{h - n - 1}{n + 1} \tag{10}$$

are simultaneously verified. Using the factorial representation of the binomial coefficients and omitting the intermediate steps for the sake of brevity, the inequality

$$5n^2 - (5h + 7)n + h^2 + 3h + 2 \geqslant 0 \tag{11}$$

is obtained from (9); the roots of the associate equation are

$$\begin{cases} n_1 = (5h + 7 - \sqrt{\Delta})/10, \\ n_2 = (5h + 7 + \sqrt{\Delta})/10, \end{cases} \tag{12}$$

where $\Delta = 5h^2 + 10h + 9$. From (11), we have

$$n_2 \leqslant n \leqslant n_1. \tag{13}$$

Analogously, from (10), we obtain the inequality

$$5n^2 - (5h - 3)n + h^2 - 2h \leqslant 0, \tag{14}$$

from which the roots

$$\begin{cases} n_1' = (5h - 3 - \sqrt{\Delta})/10 \\ n_2' = (5h - 3 + \sqrt{\Delta})/10 \end{cases} \tag{15}$$

are derived. From (14), we have

$$n_1' \leqslant n \leqslant n_2'. \tag{16}$$

Since, for $h \geqslant 2$, the inequality $n_1 < n_2'$ holds, the inequalities (13) and (16) are simultaneously verified within the interval $[n_1', n_1]$. Therefore, we have $n_1' \leqslant \hat{n}_h \leqslant n_1$. Since $n_1 - n_1' = 1$, the value

$$\hat{n}_h = [n_1'] + 1 = [n_1] \tag{17}$$

is unique, provided that n_1' (and n_1) is not an integer. If and only if n_1' is an integer is the binomial coefficient (8) maximized by two consecutive values $\hat{n}_{h,1}$ and $\hat{n}_{h,2}$ of n; that is,

$$\begin{cases} \hat{n}_{h,1} = n_1', \\ \hat{n}_{h,2} = n_1' + 1 = n_1. \end{cases} \tag{17'}$$

Now we can state the following theorem.

Theorem 5: $\hat{f}(n, k) = \left[\dfrac{5k - 8 - (5k^2 + 4)^{1/2}}{10}\right] + 1.$

Proof: The proof is derived directly from (17), (17′), and (15) after replacing h by $k - 1$ and n by m in (8). Q.E.D.

On the basis of (17′) and (15), we determine the values of k for which the quantity

$$R_k = (5k - 8 - (5k^2 + 4)^{1/2})/10$$

is integral, i.e., the values of k for which two consecutive values of m maximize $N_{k,m}$ thus yielding the following two values of $\hat{f}(n, k)$:

$$\begin{cases} \hat{f}_1(n, k) = R_k, & (18) \\ \hat{f}_2(n, k) = R_k + 1. & (18') \end{cases}$$

Theorem 6: The most probable values of $f(n, k)$ are both $\hat{f}_1(n, k)$ and $\hat{f}_2(n, k)$, if and only if $k = F_{4s}$, $s = 1, 2, \ldots$.

Proof: For R_k to be integral, the quantity $5k^2 + 4$ must necessarily be the square of an integer, i.e., the equation

$$x^2 - 5k^2 = 4 \tag{19}$$

must be solved in integers. On the basis of [7, p. 100, pp. 197-198] and by induction on r, it is seen that, if $\{x_1, k_1\}$ is a pair of positive integers x, k with minimal x satisfying (19), then all pairs of positive integers $\{x_r, k_r\}$ satisfying this equation are defined by

$$x_r \pm \sqrt{5}k_r = \frac{(x_1 \pm \sqrt{5}k_1)^r}{2^{r-1}}, \quad r = 1, 2, \ldots . \tag{20}$$

Since it is found that $x_1 = 3$ and $k_1 = 1$, from (20), we can write

$$x_r + \sqrt{5}k_r = \frac{(3 + \sqrt{5})^r}{2^{r-1}} = 2\alpha^{2r}. \tag{21}$$

From (19) and (21), we get the relation

$$(5k_r^2 + 4)^{1/2} = 2\alpha^{2r} - \sqrt{5}k_r$$

from which, squaring both sides, we obtain

$$k_r = \frac{1}{\sqrt{5}}\frac{\alpha^{4r} - 1}{\alpha^{2r}} = \frac{1}{\sqrt{5}}(\alpha^{2r} - \alpha^{-2r}) = F_{2r}.$$

Replacing k by F_{2r}, R_k reduces to $(L_{2r-1} - 4)/5$; therefore, to prove the theorem, it is sufficient to prove that, iff r is even, then the congruence $L_{2r-1} \equiv 4 \pmod 5$ holds.

Using Binet's form for L_r, we obtain

$$L_{2r-1} = \frac{1 + S}{2^{2(r-1)}},$$

where

$$S = \sum_{t=1}^{r-1}\binom{2r-1}{2t}(\sqrt{5})^{2t} = 5\sum_{t=1}^{r-1}\binom{2r-1}{2t}(\sqrt{5})^{2(t-1)}.$$

Therefore, we can write the following equivalent congruences,

$$2^{-2(r-1)}(1 + S) \equiv 4 \pmod 5,$$

$$1 + S \equiv 2^{2r} \pmod 5,$$

$$1 \equiv 2^{2r} \pmod 5,$$

which, for Fermat's little theorem, hold iff $r = 2s$, $s = 1, 2, \ldots$. Q.E.D.

4. THE INTEGERS REPRESENTABLE AS A SUM OF m_k F-ADDENDS

In this section, the set of all integers $n \in \mathcal{N}_k$ which can be represented as a sum of m_k F-addends [i.e., all integers such that $f(n, k) = m_k$] is determined.

From Theorem 2 and (1), the following corollary is immediately derived.

Corollary 1:

$$N_{k, m_k} = \begin{cases} k/2, & \text{for even } k, \\ 1, & \text{for odd } k. \end{cases}$$

The following identities are used to prove Theorems 7 and 8.

Identity 1: $\sum_{j=1}^{h} F_{2j} = F_{2h+1} - 1.$

Identity 2: $\sum_{j=1}^{h} F_{2j+1} = F_{2(h+1)} - 1.$

Identity 3: $\sum_{j=0}^{m-1} F_{2j+n} = F_{2m+n-1} - F_{n-1}.$

The proofs of Identities 1, 2, and 3 are obtained by mathematical induction and are omitted here for the sake of brevity.

Theorem 7: $f(F_k - 1) = m_k$.

Proof: (i) Even k.

For even k, we have $m_k = (k - 2)/2$; it follows that $k = 2(m_k + 1)$ and, from Identity 2,

$$F_k - 1 = F_{2(m_k+1)} - 1 = \sum_{i=1}^{m_k} F_{2i+1}.$$

(ii) Odd k.

For odd k, we have $m_k = (k - 1)/2$; it follows that $k = 2(m_k + 1)$ and, from Identity 1,

$$F_k - 1 = F_{2m_k+1} - 1 = \sum_{i=1}^{m_k} F_{2i}.$$

In both cases, $F_k - 1$ can be represented as a sum of m_k F-addends. Q.E.D.

From Theorem 7 and Corollary 1, it is evident that, for odd k, the only integer $n \in \mathcal{N}_k$ such that $f(n, k) = m_k$ is $n = F_k - 1$. Moreover, it is seen that, for even k, the integers $n \in \mathcal{N}_k$ such that $f(n, k) = m_k = (k - 2)/2$ are $k/2$ in number ($F_k - 1$ inclusive); let us denote these integers by

$$A_{k,i},\ i = 1, 2, \ldots, k/2.$$

Theorem 8: $A_{k,i} = F_k - F_{k-2i} - 1,\ i = 1, 2, \ldots, k/2.$

Proof: For a given even k, the integers $A_{k,i}$ can be obtained by means of the following procedure:

$$\begin{aligned}
A_{k,1} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + F_{k-4} + F_{k-2}\\
A_{k,2} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + F_{k-4} + (F_{k-1})\\
A_{k,3} &= F_2 + F_4 + F_6 + \cdots + F_{k-6} + (F_{k-3} + F_{k-1})\\
&\vdots\\
A_{k,k/2-2} &= F_2 + F_4 + (F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})\\
A_{k,k/2-1} &= F_2 + (F_5 + F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})\\
A_{k,k/2} &= (F_3 + F_5 + F_7 + \cdots + F_{k-5} + F_{k-3} + F_{k-1})
\end{aligned}$$

The mechanism of choice of the F-addends from two disjoint subsets of $\mathscr{F}_k$ [namely, $\{F_{2t}\}$ and $\{F_{2t+1}\}$, $t = 1, 2, \ldots, (k-2)/2$] illustrated in the previous table yields the following expression of $A_{k,i}$,

$$A_{k,i} = \sum_{r=0}^{k/2-i-1} F_{2+2r} + \sum_{s=0}^{i-2} F_{k-2i+2s+3},$$

from which, by virtue of Identity 3, we obtain

$$\begin{aligned} A_{k,i} &= F_{2(k/2-i)+1} - F_1 + F_{2(i-1)+k-2i+2} - F_{k-2i+2} \\ &= F_{k-2i+1} - 1 + F_k - F_{k-2i+2} = F_k - F_{k-2i} - 1. \quad \text{Q.E.D.} \end{aligned}$$

The following corollary is derived from Theorem 8.

Corollary 2: $A_{k,1} = F_{k-1} - 1,$ (22)

$A_{k,2} = L_{k-2} - 1,$ (23)

$A_{k,k/2} = F_k - 1.$ (24)

Proof: Identities (22) and (24) are obtained directly from Theorem 8. Identity (23) requires some manipulations; that is,

$$\begin{aligned} A_{k,2} &= F_k - F_{k-4} - 1 = F_k - (5F_k - 3F_{k+1}) - 1 \\ &= -F_k + 3(F_{k+1} - F_k) - 1 = -F_k + 3F_{k-1} - 1 \\ &= 2F_{k-1} - F_{k-2} - 1 = F_{k-1} + F_{k-3} - 1 = L_{k-2} - 1. \quad \text{Q.E.D.} \end{aligned}$$

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