MATRIX AND OTHER SUMMATION TECHNIQUES FOR PELL POLYNOMIALS

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1. INTRODUCTION

Pell polynomials $P_n(x)$ and *Pell-Lucas polynomials* $Q_n(x)$ are defined in [7], [9], and [10] by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x), \quad P_0(x) = 0, \; P_1(x) = 1, \tag{1.1}$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x), \quad Q_0(x) = 2, \; Q_1(x) = 2^x, \tag{1.2}$$

with integer n unrestricted.

Equation (1.1) may be written in the form

$$P_r(x) = \{P_{r+1}(x) - P_{r-1}(x)\}/2x. \qquad (1.1)'$$

Binet forms are

$$P_n(x) = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{1.3}$$

and

$$Q_n(x) = \alpha^n + \beta^n, \tag{1.4}$$

where α and β are the roots of the characteristic equation of (1.1) and (1.2), namely,

$$t^2 - 2xt - 1 = 0 \tag{1.5}$$

so that

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \quad \text{with } \alpha + \beta = 2x, \; \alpha\beta = -1, \; \alpha - \beta = 2\sqrt{x^2 + 1}. \tag{1.6}$$

Explicit summation representations for $P_n(x)$ and $Q_n(x)$, and relations among them, are established in [7], [9], and [10].

Emphasis in this paper will be given to matrix methods so we introduce the matrix P which generates Pell polynomials and many of their properties ([7], [9]). Historical information about the background of this matrix is provided in [9].

Let

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \tag{1.7}$$

so that, by induction,

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \tag{1.8}$$

Hence,

$$\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{1.9}$$

and so

$$P_n(x) = [1 \quad 0]\, P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.10}$$

From [7, (2.1)], we deduce

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix} \tag{1.11}$$

and

$$Q_n(x) = [1 \quad 0]\, P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{1.12}$$

Although some summation formulas for $P_n(x)$ and $Q_n(x)$ are recorded in [7], it is thought desirable to investigate the summation problem more fully. Initially, some well-established techniques are utilized to produce simple summations. More complicated techniques are derived to achieve a higher degree of completeness.

As an example of the usage of the matrix (and determinant) approach, we demonstrate the Simson formula for Pell polynomials, [7, (2.5)], namely,

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n, \tag{1.13}$$

which may, of course, be established by means of the Binet form (1.3).

More generally in the first instance, consider

$$P_n^2(x) - P_{n+r}(x)P_{n-r}(x) = \begin{vmatrix} P_n(x) & P_{n+r}(x) \\ P_{n-r}(x) & P_n(x) \end{vmatrix} \tag{1.14}$$

$$= \left| \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^{n-r} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|$$

... by (1.8), [7, (3.14)]

$$= \begin{vmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{vmatrix} \left| P^{n-r} \right| \begin{vmatrix} 1 & P_{r+1}(x) \\ 0 & P_r(x) \end{vmatrix}, \text{ by (1.9)}, = (-1)^{n-r} P_r^2(x).$$

Putting $r = 1$ in this generalized Simson formula, we obtain the Pell-analogue (1.13) of the Simson formula for Fibonacci numbers.

Because of its importance and subsequent use, we append the difference equation [7, (3.28)]

$$P_{mn+r}(x) = \begin{cases} P_m(x)Q_{(n-1)m+r}(x) + (-1)^m P_{(n-2)m+r}(x) \\ Q_m(x)P_{(n-1)m+r}(x) + (-1)^{m-1}P_{(n-2)m+r}(x) \end{cases} \tag{1.15}$$

and the Pell-Lucas analogue [7, (3.29)]

$$Q_{mn+r}(x) = Q_m(x)Q_{(n-1)m+r}(x) + (-1)^{m-1}Q_{(n-2)m+r}(x). \tag{1.16}$$

A result needed in Section 8, which is not specifically given in [7], is

$$Q_n(x)Q_{n+1}(x) - 4(x^2 + 1)P_n(x)P_{n+1}(x) = 4x(-1)^n, \tag{1.17}$$

which may be proved by using (1.3) and (1.4).

2. SOME SUMMATION TECHNIQUES

A. Consider the *series of matrices* [cf. (1.8)]

$$A = I + P + P^2 + \cdots + P^{n-2} + P^{n-1}.$$

Then

$$PA = P + P^2 + P^3 + \cdots + P^{n-1} + P^n,$$

whence

$$\begin{aligned} (P - I)A &= P^n - I \\ A &= (P - I)^{-1}(P^n - I) \\ &= \frac{1}{2x}\begin{bmatrix} 1 & 1 \\ 1 & 1 - 2x \end{bmatrix}\begin{bmatrix} P_{n+1}(x) - 1 & P_n(x) \\ P_n(x) & P_{n-1}(x) - 1 \end{bmatrix} \quad \text{by (1.8)} \\ &= \frac{1}{2x}\begin{bmatrix} P_{n+1}(x) + P_n(x) - 1 & P_n(x) + P_{n-1}(x) - 1 \\ P_n(x) + P_{n-1}(x) - 1 & P_{n-2}(x) + P_{n-1}(x) + 2x - 1 \end{bmatrix}. \end{aligned}$$

Now

$$\sum_{r=1}^{n} P_r(x) = [1 \quad 0]A\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ by (1.10)}.$$

Hence

$$\sum_{r=1}^{n} P_r(x) = (P_{n+1}(x) + P_n(x) - 1)/2x. \tag{2.1}$$

B. Using the Binet form (1.4), we have

$$\sum_{r=1}^{n} Q_r(x) = \sum_{r=1}^{n} (\alpha^r + \beta^r)$$

which, with the application of the summation formula for a geometric series and the properties of α and β, reduces to

$$\sum_{r=1}^{n} Q_r(x) = (Q_{n+1}(x) + Q_n(x) - 2x - 2)/2x. \tag{2.2}$$

Clearly, the matrix technique A could be used here also.

C. Next, we use *difference equations* derived from the recurrence relation (1.1), namely,

$$\begin{aligned} 2xP_1(x) &= P_2(x) - P_0(x) \\ 2xP_3(x) &= P_4(x) - P_2(x) \\ &\cdots\cdots\cdots\cdots \\ 2xP_{2n-1}(x) &= P_{2n}(x) - P_{2n-2}(x) \end{aligned}$$

whence, on addition and simplification,

$$\sum_{r=1}^{n} P_{2r-1}(x) = P_{2n}(x)/2x. \tag{2.3}$$

Summation formulas for

$$\sum_{r=1}^{n} P_{2r}(x), \quad \sum_{r=1}^{n} Q_{2r-1}(x), \quad \text{and} \quad \sum_{r=1}^{n} Q_{2r}(x)$$

are given in [9], as indeed are (2.1), (2.2), and (2.3).

D. Fourthly, we utilize an extension of technique C. In this method, our aim is to find sums of series of Pell polynomials with subscripts in arithmetic progression.

Let

$$S_1 = \sum_{i=1}^{n} P_{im}(x), \quad S_2 = \sum_{i=1}^{n} P_{im-1}(x),$$

$$S_3 = \sum_{i=1}^{n} P_{im-2}(x), \quad \ldots, \quad S_m = \sum_{i=1}^{n} P_{im-(m-1)}(x). \tag{2.4}$$

Then, the set of equations connecting the members of $\{S_i\}$ in (2.4) may be shown to be:

$$\begin{cases} 2xS_1 + S_2 \cdots\cdots\cdots\cdots - S_m = P_{nm+1}(x) - P_1(x) \\ -S_1 + 2xS_2 + S_3 \cdots\cdots\cdots\cdots = 0 \\ \qquad -S_2 + 2xS_3 + S_4 \cdots\cdots\cdots = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots -S_{m-2} + 2xS_{m-1} + S_m = 0 \\ S_1 \cdots\cdots\cdots\cdots -S_{m-1} + 2xS_m = P_{nm}(x) - P_0(x) \end{cases} \tag{2.5}$$

Next, write:

$$\mathscr{E}_1 = \begin{bmatrix} 2x \\ -1 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \mathscr{E}_2 = \begin{bmatrix} 1 \\ 2x \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \mathscr{E}_3 = \begin{bmatrix} 0 \\ 1 \\ 2x \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \ldots, \ \mathscr{E}_m = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 1 \\ 2x \end{bmatrix} \tag{2.6}$$

$$\mathscr{F} = \begin{bmatrix} P_{nm+1}(x) - P_1(x) \\ 0 \\ 0 \\ \vdots \\ 0 \\ P_{nm}(x) - P_0(x) \end{bmatrix},$$

where $\mathscr{E}_i$ and $\mathscr{F}$ are $m \times 1$ matrices.

Denote by e_{ij} the element in the i^{th} row of $\mathscr{E}_j$.

Matrices in (2.6) are then defined by:

$$\begin{cases} e_{ii} = 2x & \text{for } i = 1, 2, \ldots, m \\ e_{1m} = -1 & \\ e_{m1} = 1 & \\ e_{i,i+1} = 1 & \text{for } i = 1, 2, \ldots, m-1 \\ e_{i-1,i} = -1 & \text{for } i = 2, 3, \ldots, m \\ e_{ij} = 0 & \text{otherwise.} \end{cases} \tag{2.6$'$}$$

All the entries in $\mathscr{F}$, except those in the first and last rows, are zero.

Write

$$\psi_m(x) = \left| \mathscr{E}_1 \ \vdots \ \mathscr{E}_2 \ \vdots \ \cdots \ \vdots \ \mathscr{E}_m \right|. \tag{2.7}$$

Designate by $\psi_m^{(i)}(x)$ the determinant obtained from $\psi_m(x)$ in (2.7) by replacing the i^{th} column by $\mathscr{F}$ in (2.6).

Cramer's Rule then gives the solution of the system of equations (2.5) as

$$S_i = \frac{\psi_m^{(i)}(x)}{\psi_m(x)}. \tag{2.8}$$

Comparing this result with (2.10) below leads us to the identity [compare (3.15, (3.16)]

$$\psi_m(x) = Q_m(x) - 1 + (-1)^{m+1}, \tag{2.9}$$

which may be proved by induction.

One may use whichever of the above techniques, A-D, is most appropriate to the occasion.

This brief illustration of four simple techniques is by no means exhaustive. Other methods will be suggested later.

More generally, let

$$\mathscr{P} = P_{m+k}(x) + P_{2m+k}(x) + P_{3m+k}(x) + \cdots + P_{mn+k}(x).$$

$$\therefore \quad -Q_m(x)\mathscr{P} = -Q_m(x)P_{m+k}(x) - Q_m(x)P_{2m+k}(x) - Q_m(x)P_{3m+k}(x) - \cdots - Q_m(x)P_{mn+k}(x)$$

$$(-1)^m\mathscr{P} = (-1)^m P_{m+k} + (-1)^m P_{2m+k}(x) + (-1)^m P_{3m+k}(x) + \cdots + (-1)^m P_{mn+k}(x).$$

Add and use equation (1.15) to obtain, with care,

$$\sum_{r=1}^{n} P_{mr+k}(x) = \frac{(-1)^m\{P_{mn+k}(x) - P_k(x)\} - \{P_{m(n+1)+k}(x) - P_{m+k}(x)\}}{1 - Q_m(x) + (-1)^m}. \tag{2.10}$$

Similarly,

$$\sum_{r=1}^{n} Q_{mr+k}(x) = \frac{(-1)^m\{Q_{mn+k}(x) - Q_k(x)\} - \{Q_{m(n+1)+k}(x) - Q_{m+k}(x)\}}{1 - Q_m(x) + (-1)^m}. \tag{2.11}$$

Results (2.10) and (2.11) could be obtained laboriously by other means, e.g., by using the Binet form or the matrix P.

Various specializations of (2.10) and (2.11) appearing in [9] are of interest, as, e.g.,

$$\sum_{r=1}^{n} P_{3r}(x) = \{P_{3n+3}(x) + P_{3n}(x) - P_3(x)\}/Q_3(x). \tag{2.12}$$

Several interesting simplifications arise when $m = 4a$ and $m = 4a + 2$, e.g., after manipulation,

$$\sum_{r=1}^{n} P_{4ar+k}(x) = P_{2a(n+1)+k}(x)P_{2an}(x)/P_{2a}(x). \tag{2.13}$$

Details are given in [9].

3. DETERMINANTAL GENERATION

Following the ideas and notation in [7], let us define the determinants of order n below, where d_{ij} is the entry in row i and column j:

$$\Delta_{n,m}(x) : \begin{cases} d_{ii} = Q_m(x) & i = 1, 2, \ldots, n \\ d_{i,i+1} = 1 & i = 1, 2, \ldots, n-1 \\ d_{i,i-1} = (-1)^m & i = 2, \ldots, n \\ d_{ij} = 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

$\delta_{n,m}(x)$: as for $\Delta_{n,m}(x)$ except that $d_{i,i+1} = -1$, $d_{i,i-1} = -(-1)^m$. (3.2)

$\Delta^*_{n,m}(x)$: as for $\Delta_{n,m}(x)$ except that $d_{12} = 2$. (3.3)

$\delta^*_{n,m}(x)$: as for $\delta_{n,m}(x)$ except that $d_{12} = -2$. (3.4)

Using the method of induction, we can establish that

$$\Delta_{n,m}(x) = P_{(n+1)m}(x)/P_m(x). \tag{3.5}$$

When $m = 1$, (3.5) becomes equation (5.5) in [7]. For $m = k + 1$, we use equation (1.15) to validate (3.5).

Similarly, we demonstrate with the aid of (1.16) that

$$\delta_{n,m}(x) = P_{(n+1)m}(x)/P_m(x). \tag{3.6}$$

In a similar vein, we may show that

$$\Delta^*_{n,m}(x) = Q_{mn}(x) \tag{3.7}$$

and

$$\delta^*_{n,m}(x) = Q_{mn}(x). \tag{3.8}$$

Suitable expansion down columns or along rows yields:

$$\Delta_{n,m}(x) = Q_m(x)\Delta_{n-1,m}(x) + (-1)^{m+1}\Delta_{n-2,m}(x); \tag{3.9}$$

$$\delta_{n,m}(x) = Q_m(x)\delta_{n-1,m}(x) + (-1)^{m+1}\delta_{n-2,m}(x); \tag{3.10}$$

$$\begin{aligned}\Delta^*_{n,m}(x) &= Q_m(x)\Delta^*_{n-1,m}(x) + (-1)^{m+1}\Delta^*_{n-2,m}(x)\\ &= Q_m(x)\Delta_{n-1,m}(x) + 2(-1)^{m+1}\Delta_{n-2,m}(x);\end{aligned} \tag{3.11}$$

$$\begin{aligned}\delta^*_{n,m}(x) &= Q_m(x)\delta^*_{n-1,m}(x) + (-1)^{m+1}\delta^*_{n-2,m}(x)\\ &= Q_m(x)\delta_{n-1,m}(x) + 2(-1)^{m+1}\delta_{n-2,m}(x).\end{aligned} \tag{3.12}$$

Putting $m = 1$ in (3.5)-(3.8), and in (3.9) and (3.11), we readily obtain the equations (5.5)-(5.8), and (5.9) and (5.10), respectively, in [7]. Moreover, $\Delta_{n,1}(1) = \delta_{n,1}(1) = P_{n+1}$ and $\Delta^*_{n,1}(1) = \delta^*_{n,1}(1) = Q_n$, where P_{n+1} and Q_n are *Pell numbers* and *Pell-Lucas numbers*, respectively, occurring when $x = 1$.

Variations, though small, of the determinants (3.1)-(3.4) above and of their specializations when $m = 1$, as given in [7], are used in [9] to obtain (3.5)-(3.12). Mahon, in [9], conceived these determinants with some complex entries as extensions of a determinant utilized in [2] and [8].

Next, consider the determinant $\omega_{n,m}(x)$ of order n defined by

$$\omega_{n,m}(x) : \begin{cases} d_{ii} = Q_m(x) & i = 1, 2, \ldots n \\ d_{i,i+1} = -1 & i = 1, 2, \ldots, n-1 \\ d_{i,i-1} = -(-1)^m & i = 2, 3, \ldots, n \\ d_{n1} = (-1)^m & \\ d_{1n} = 1 & \\ d_{ij} = 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Careful evaluation of this determinant, with appeal to (3.8) and (3.12) gives us

$$\omega_{n,m}(x) = Q_{mn}(x) + (-1)^m + (-1)^{m(n-1)}. \tag{3.14}$$

In particular, when $m = 1$, and writing $\omega_n(x) \equiv \omega_{n,1}(x)$, we get:

$$\omega_n(x) = Q_n(x) - 1 + (-1)^{n+1}; \tag{3.15}$$

$$\omega_{2n-1}(x) = Q_{2n-1}(x); \tag{3.16}$$

$$\omega_{4n}(x) = 4(x^2 + 1)P_{2n}^2(x), \text{ by equation (2.18) in [7];} \tag{3.17}$$

$$\omega_{4n+2}(x) = Q_{2n+1}^2(x); \tag{3.18}$$

where, to obtain (3.18), we may use result (3.25) in [7] in which n and r are both replaced by $2n + 1$ [$Q_0(x) = 2$].

Observe that $-\omega_n(x)$ in (3.15) is precisely the form of the denominator in (2.10) and (2.11). [Cf. (2.9).] Indeed, it was in this context that the need to investigate the determinants $\omega_n(x)$ arose.

4. ALTERNATING AND RELATED SERIES

To avoid tedium and to save some space, we will as a rule hereafter merely give the results of the more important summations which we desire to record. Some of the proofs are quite difficult.

$$\sum_{r=1}^{n} rP_r(x) = [nxP_{n+1}(x) + \{(n - 1)x - 1\}P_n(x) - P_{n-1}(x) + 1]/2x^2. \tag{4.1}$$

Proving this is straightforward. From (1.1)′, we have

$$2xP_1(x) = P_2(x) - P_0(x).$$

Multiply this by 2, 3, ..., n in turn, add, and use (2.1). Then (4.1) results.

Similarly, we establish

$$\sum_{r=1}^{n} rQ_r(x) = [nxQ_{n+1}(x) + \{(n - 1)x - 1\}Q_n(x) - Q_{n-1}(x) + 2]/2x^2; \tag{4.2}$$

$$\sum_{r=1}^{n} (-1)^r rP_r(x) = [(-1)^n nxP_{n+1}(x) + (-1)^{n-1}P_n(x)\{(n - 1)x + 1\} + (-1)^n P_{n-1}(x) - 1]/2x^2; \tag{4.3}$$

$$\sum_{r=1}^{n} (-1)^r rQ_r(x) = [(-1)^n nxQ_{n+1}(x) + (-1)^{n-1}Q_n(x)\{(n - 1)x + 1\} + (-1)^n Q_{n-1}(x) - Q_1(x) + Q_0(x)(1 + x)]/2x^2. \tag{4.4}$$

More generally, suppose we write

$$F(n, x, y) = \sum_{r=1}^{n} P_{mr+k}(x)y^r \tag{4.5}$$

and

$$G(n, x, y) = \sum_{r=1}^{n} Q_{mr+k}(x)y^r. \tag{4.6}$$

Now use (1.15) and (1.16) for $P_{m+k}(x)$, $P_{2m+k}(x)$, ..., $P_{mn+k}(x)$ and $Q_{m+k}(x)$, $Q_{2m+k}(x)$, ..., $Q_{mn+k}(x)$, add and obtain explicit expressions for $F(n, x, y)$ and $G(n, x, y)$. Details of these calculations are left to the reader. If we then put $y = 1$, we derive formulas for

$$\sum_{r=1}^{n} P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n} Q_{mr+k}(x).$$

On the other hand, $y = -1$ leads to formulas for

$$\sum_{r=1}^{n}(-1)^r P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n}(-1)^r Q_{mr+k}(x).$$

Differentiating with respect to y in (4.5) and (4.6) gives

$$\sum_{r=1}^{n} rP_{mr+k}(x) = F'(n, x, 1), \tag{4.7}$$

$$\sum_{r=1}^{n} rQ_{mr+k}(x) = G'(n, x, 1), \tag{4.8}$$

$$\sum_{r=1}^{n}(-1)^{r-1} rP_{mr+k}(x) = F'(n, x, -1), \tag{4.9}$$

$$\sum_{r=1}^{n}(-1)^{r-1} rQ_{mr+k}(x) = G'(n, x, -1), \tag{4.10}$$

in which the prime denotes the derivative with respect to y. When $m = 1$, $k = 0$ in (4.7)-(4.10), (4.1)-(4.4) occur.

Next, consider $P_1(x) = \{P_2(x) - P_0(x)\}/2x$ from the recurrence (1.1)′. Multiply this equation by $2^2, 3^2, \ldots, n^2$ in turn, add, and use (4.1). Then

$$\sum_{r=1}^{n} r^2 P_r(x) = [2n^2x^2P_{n+1}(x) + 2(n-1)x\{(n-1)x - 2\}P_n(x) - 4\{(n-2)x - 1\}P_{n-1}(x) + 4P_{n-2}(x) - 4]/4x^3. \tag{4.11}$$

Similarly,

$$\sum_{r=1}^{n} r^2 Q_r(x) = [2n^2x^2Q_{n+1}(x) + 2(n-1)x\{(n-1)x - 2\}Q_n(x) - 4\{(n-2)x - 1\}Q_{n-1}(x) + 4Q_{n-2}(x) - 4x^2 - 8]/4x^3, \tag{4.12}$$

$$\sum_{r=1}^{n}(-1)^r r^2 P_r(x) = [(-1)^n 2x^2n^2P_{n+1}(x) + (-1)^{n-1}2x(n-1)P_n(x)\{x(n-1) + 2\} + 4(-1)^{n-2}P_{n-1}(x)\{1 + (n-2)x\} + 4(-1)^{n-1}P_{n-2}(x) - 4]/4x^3, \tag{4.13}$$

$$\sum_{r=1}^{n}(-1)^r r^2 Q_r(x) = [(-1)^n 2x^2n^2Q_{n+1}(x) + (-1)^{n-1}2x(n-1)Q_n(x)\{x(n-1) + 2\} + 4(-1)^{n-2}Q_{n-1}(x)\{1 + (n-2)x\} + 4(-1)^{n-1}Q_{n-2}(x) + 4x^2 + 8]/4x^3. \tag{4.14}$$

Other methods for obtaining the above results in this section are available, for example the difference equation technique employed in [9], although this involves a great deal of complicated algebraic manipulation. Of the various approaches open to us for obtaining the summations, perhaps the most powerful and most appealing procedure is that using difference equations. Indeed, by employing one such difference equation, Mahon [9] has found formulas involving the generalized summations

$$\sum_{r=1}^{n} r^t P_{mr+k}(x) \quad \text{and} \quad \sum_{r=1}^{n} r^t Q_{mr+k}(x),$$

but the results are not a pretty sight!.

To give a flavor for these difference equations, we record one used in the construction of the formula (4.11) by this method, namely,

$$(r+1)^2 P_{m(r+1)+k}(x) - Q_m(x) r^2 P_{mr+k}(x) + (-1)^m (r-1)^2 P_{m(r-1)+k}(x)$$
$$= 2rP_m(x)Q_{mr+k}(x) + Q_m(x)P_{mr+k}(x).$$

Many similar complicated results are given in [9].

To conclude this section, we append some sums of cubes of $P_n(x)$ and $Q_n(x)$ obtained with the aid of the Binet formulas (1.3) and (1.4).

$$\sum_{r=1}^{n} P_r^3(x) = [P_{3n+3}(x) + P_{3n}(x) - 3(4x^2+3)\{(-1)^n (P_{n+1}(x) - P_n(x)\} + 8(x^2+1)]/4(x^2+1)Q_3(x). \tag{4.16}$$

$$\sum_{r=1}^{n} Q_r^3(x) = [Q_{3n+3}(x) + Q_{3n}(x) - Q_3(x) - Q_0(x) + 3(4x^2+3)\{(-1)^n Q_{n+1}(x) - Q_n(x)) - Q_1(x) + Q_0(x)\}]/Q_3(x). \tag{4.17}$$

$$\sum_{r=1}^{n} (-1)^r P_r^3(x) = [(-1)^n \{P_{3n+3}(x) - P_{3n}(x)\} - P_3(x) - 3(4x^2+3)\{P_{n+1}(x) + P_n(x) - 1\}]/4(x^2+1)Q_3(x). \tag{4.18}$$

$$\sum_{r=1}^{n} (-1)^r Q_r^3(x) = [(-1)^n \{Q_{3n+3}(x) - Q_{3n}(x)\} - \{Q_3(x) - Q_0(x)\} + 3(4x^2+3)\{Q_{n+1}(x) + Q_n(x) - Q_1(x) - Q_0(x)\}]/Q_3(x). \tag{4.19}$$

5. SERIES OF SQUARES AND PRODUCTS OF $P_n(x)$ AND $Q_n(x)$

Multiply both sides of (1.1)′ by $P_r(x)$ and add. Then

$$\sum_{r=1}^{n} P_r^2(x) = P_{n+1}(x)P_n(x)/2x. \tag{5.1}$$

Similarly,

$$\sum_{r=1}^{n} Q_r^2(x) = \{Q_{n+1}(x)Q_n(x) - 4x\}/2x. \tag{5.2}$$

Again, in this development, the method of difference equations has general applicability. For instance, after much algebraic maneuvering, one can obtain the difference equation appropriate to (5.1), namely,

$$P_{n+1}^2(x) - (4x^2+2)P_n^2(x) + P_{n-1}^2(x) = 2(-1)^n. \tag{5.1a}$$

More generally, difference equations can be applied to find formulas for

$$\sum_{r=1}^{n} P_{mr+k}^2(x) \quad \text{and} \quad \sum_{r=1}^{n} Q_{mr+k}^2(x).$$

For the former summation, for instance, the difference equation is

$$P_{m(r+1)+k}^2(x) - Q_{2m}(x)P_{mr+k}(x) + P_{m(r-1)+k}^2(x) = 2P_m^2(x)(-1)^{mr+k}, \tag{5.1b}$$

which reduces to the simpler form (5.1a) when $m = 1$, $k = 0$ (and r is replaced by n).

If we multiply both sides of (1.1) by $P_{r-1}(x)$ and add, then, by Simson's formula (1.14),

$$\sum_{r=1}^{n} P_{r-1}(x)P_r(x) = \{P_n^2(x) - \tfrac{1}{2}(1 - (-1)^n)\}/2x. \tag{5.3}$$

Similarly,

$$\sum_{r=1}^{n} Q_{r-1}(x)Q_r(x) = \{Q_n^2(x) - 4 + 2(x^2 + 1)(1 - (-1)^n)\}/2x. \tag{5.4}$$

Alternating series may be summed using (1.1)′. First, write

$$D = \sum_{r=1}^{n} (-1)^r P_r^2(x), \quad E = \sum_{r=1}^{n} (-1)^{r-1} P_{r-1}(x)P_r(x).$$

Then, multiplying both sides of (1.1)′ by $(-1)^r P_r(x)$ and adding gives

$$2xD - 2E = (-1)^n P_n(x)P_{n+1}(x) \;\ldots\ldots\ldots\; \text{(i)}.$$

Next, multiplying both sides of (1.1)′ by $(-1)^{r-1}P_{r-1}(x)$ and adding gives

$$2D + 2xE = (-1)^n P_n^2(x) - n \;\ldots\ldots\ldots\; \text{(ii)}.$$

Solve (i) and (ii), and use (2.1) and (2.3) in [7] to obtain

$$\sum_{r=1}^{n} (-1)^r P_r^2(x) = \{(-1)^n Q_{n+1}(x)P_n(x) - 2n\}/4(x^2 + 1) \tag{5.5}$$

and

$$\sum_{r=1}^{n} (-1)^{r-1} P_{r-1}(x)P_r(x) = \{(-1)^{n+1} P_{2n}(x) - 2nx\}/4(x^2 + 1). \tag{5.6}$$

Similarly,

$$\sum_{r=1}^{n} (-1)^r Q_r^2(x) = (-1)^n Q_n(x)P_{n+1}(x) + 2(n - 1) \tag{5.7}$$

and

$$\sum_{r=1}^{n} (-1)^{r-1} Q_{r-1}(x)Q_r(x) = 2nx + (-1)^{n+1} P_{2n}(x). \tag{5.8}$$

Now multiply both sides of (1.1)′ by $(-1)^r rP_r(x)$ and sum. Write

$$D_1 = \sum_{r=1}^{n} (-1)^r rP_r^2 \quad \text{and} \quad E_1 = \sum_{r=1}^{n} (-1)^r (2r - 1)P_{r-1}(x)P_r(x).$$

Then

$$2xD_1 + E_1 = n(-1)^n P_n(x)P_{n+1}(x) \;\ldots\ldots\ldots\; \text{(iii)},$$

$$4D_1 - 2xE_1 = (-1)^n(2n + 1)P_n^2(x) - n^2 \;\ldots\ldots\; \text{(iv)},$$

where, in (iv), we have multiplied both sides of (1.1)′ by

$$(-1)^{r-1}(2r - 1)P_{r-1}(x)$$

and summed.

Solve (iii) and (iv) to obtain

$$\sum_{r=1}^{n}(-1)^r rP_r^2(x) = [(-1)^n P_n(x)\{nQ_{n+1}(x) + P_n(x)\} - n^2]/4(x^2+1) \tag{5.9}$$

and

$$\sum_{r=1}^{n}(-1)^{r-1}(2r-1)P_{r-1}(x)P_r(x) = [2(-1)^n P_n(x)(xP_n(x) - nQ_n(x)) - 2n^2x]/4(x^2+1). \tag{5.10}$$

Similarly,

$$\sum_{r=1}^{n}(-1)^r rQ_r^2(x) = (-1)^n[nQ_n(x)P_{n+1}(x) + P_n^2(x)] + n^2 \tag{5.11}$$

and

$$\sum_{r=1}^{n}(-1)^{r-1}(2r-1)Q_{r-1}(x)Q_r(x) = 2(-1)^n P_n(x)[xP_n(x) - nQ_n(x)] + 2n^2x. \tag{5.12}$$

Formulas for

$$\sum_{r=1}^{n}(-1)^r P_{mr+k}^2(x) \quad \text{and} \quad \sum_{r=1}^{n}(-1)^r Q_{mr+k}^2(x)$$

may be established by employing appropriate difference equations, e.g., (5.1b) in the first case.

6. COMBINATORIAL SUMMATION IDENTITIES FOR $P_n(x)$ AND $Q_n(x)$

Binomial coefficient factors associated with summations involving $P_n(x)$ and $Q_n(x)$ may be introduced to yield some useful formulas. The techniques for deriving these formulas are varied. Some approaches are indicated below.

Binet formulas (1.3) and (1.4) may be used to derive the following, for which proofs may be found in [9]:

$$\sum_{k=0}^{2n}\binom{2n}{k}P_{k+j}^2(x) = 4^{n-1}(x^2+1)^{n-1}Q_{2n+2j}(x); \tag{6.1}$$

$$\sum_{k=0}^{2n}\binom{2n}{k}Q_{k+j}^2(x) = 4^n(x^2+1)^n Q_{2n+2j}(x); \tag{6.2}$$

$$\sum_{k=0}^{2n}\binom{2n+1}{k}P_{k+j}^2(x) = 4^n(x^2+1)^n P_{2n+2j+1}(x); \tag{6.3}$$

$$\sum_{k=0}^{2n}\binom{2n+1}{k}Q_{k+j}^2(x) = 4^{n+1}(x^2+1)^{n+1}P_{2n+2j+1}(x). \tag{6.4}$$

A considerable number of combinatorial identities relating to $P_n(x)$ and $Q_n(x)$ may be determined. Among these are the general explicit expressions (developments of ideas for Fibonacci numbers in [8]—see also [3]).

$$P_{rn}(x) = \left\{\sum_{k=0}^{[(n-1)/2]}(-1)^{k(r-1)}\binom{n-1-k}{k}Q_r^{n-1-2k}(x)\right\}P_r(x) \tag{6.5}$$

and

$$Q_{rn}(x) = \sum_{k=0}^{[n/2]}(-1)^{k(r-1)}\frac{n}{n-k}\binom{n-k}{k}Q_r^{n-2k}(x), \; n \neq 0. \tag{6.6}$$

Proofs of (6.5) and (6.6) are by the method of mathematical induction.

Putting $r = 1$ in (6.5) and (6.6), we deduce the explicit expressions for $P_n(x)$ and $Q_n(x)$ given in [7] as equations (2.15) and (2.16), respectively. Other summation formulas for $P_n(x)$ are given in [9], where, further, combinatorial expressions are obtained for $P_{(2i+1)r+k}(x)$, $P_{2ir+k}(x)$, $Q_{(2i+1)r+k}(x)$, and $Q_{2ir+k}(x)$.

Bergum and Hoggatt, in [1], found expressions for sums of numbers of recurrence sequences as products of these sequences. It is possible to apply their methods to polynomials.

Two examples of this type of result are herewith given, while many others are derived in [9].

$$\sum_{i=0}^{2^j-1} P_{n+4ki}(x) = P_{n+2(2^j-1)k}(x) \prod_{i=1}^{j} Q_{2^i k}(x) \qquad (k \geqslant 1). \tag{6.7}$$

$$\sum_{i=0}^{2^j-1} Q_{n+(2i-1)k}(x) = Q_{n+2(2^{j-1}-1)k}(x) \prod_{i=0}^{j-1} Q_{2^i k}(x) \qquad (k \text{ even}). \tag{6.8}$$

To establish (6.7), we need equation (3.22) in [7], whereas (6.8) requires (3.23) in [7] together with the result for $Q_n(x)$ corresponding to (6.7) for $P_n(x)$, namely, (6.7) with $P_n(x)$ replaced by $Q_n(x)$.

7. MATRIX SUMMATION METHODS

In Section 1, the matrix P was used to obtain sums of series in which the terms contain Pell polynomials of degree one. Since the particular methods employed there were not especially convenient, we turn our attention to a more fruitful matrix approach, developing an idea expounded in [6]. Applying the Cayley-Hamilton theorem to the matrix P in (1.7), we have

$$P^2 = 2xP + I \tag{7.1}$$

whence

$$P^{2n+j} = (2xP + I)^n P^j. \tag{7.2}$$

Equating appropriate elements on both sides with the aid of (1.8), we obtain the combinatorial summations

$$P_{2n+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r P_{r+j}(x) \qquad [2x = P_2(x)] \tag{7.3}$$

and

$$P_{2n+1+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r P_{r+1+j}(x). \tag{7.4}$$

Post-multiplying both sides of (7.2) by the column vector $[2x \quad 2]^T$ (the transpose of the corresponding row vector), and appealing to (1.11), we find, on equating appropriate elements, that

$$Q_{2r+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r Q_{r+j}(x) \tag{7.5}$$

and

$$Q_{2n+1+j}(x) = \sum_{r=0}^{n} \binom{n}{r} (2x)^r Q_{r+1+j}(x). \tag{7.6}$$

Next, consider

$$\sum_{k=0}^{2n}\binom{2n}{k}P^{2k+r} = P^r(P^2 + I)^{2n}$$
$$= P^r\{2(xP + I)\}^{2n} \qquad \text{by (7.1)}$$
$$= 2^{2n}P^r(x^2P^2 + 2xP + I)^n$$
$$= 2^{2n}(x^2 + 1)^n P^{2n+r} \qquad \text{by (7.1) again,}$$

whence

$$\sum_{k=0}^{2n}\binom{2n}{k}P_{2k+r}(x) = 2^{2n}(x^2 + 1)^n P_{2n+r}(x). \tag{7.7}$$

Likewise, from (1.11),

$$\sum_{k=0}^{2n}\binom{2n}{k}Q_{2k+r}(x) = 2^{2n}(x^2 + 1)^n Q_{2n+r}(x). \tag{7.8}$$

Similarly,

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}P_{2k+r}(x) = 2^{2n}(x^2 + 1)^n Q_{2n+r+1}(x) \tag{7.9}$$

and

$$\sum_{k=0}^{2n+1}\binom{2n+1}{k}Q_{2k+r}(x) = 2^{2n+2}(x^2 + 1)^{n+1}P_{2n+r+1}(x). \tag{7.10}$$

From (7.1) it follows, since $P_2(x) = 2x$, $P_3(x) = 4x^2 + 1$, that

$$P^3 = P_3(x)P + P_2(x)I, \tag{7.11}$$

whence, after calculation,

$$P_{3n+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)P_{n-r+j}(x). \tag{7.12}$$

Since

$$P^{3n+j}\begin{bmatrix}2x\\2\end{bmatrix} = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)P^{n-r+j}\begin{bmatrix}2x\\2\end{bmatrix}, \tag{7.13}$$

then

$$Q_{3n+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_3^{n-r}(x)P_2^r(x)Q_{n-r+j}(x). \tag{7.14}$$

Note in (7.12) and (7.14) the emergence of extra terms in the summation, a fact which was hidden in (7.3) and (7.5) by $P_1(x) = 1$.

More generally, one can show that

$$P_{kn+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_k^{n-r}(x)P_{k-1}^r(x)P_{n-r+j}(x) \tag{7.15}$$

and

$$Q_{kn+j}(x) = \sum_{r=0}^{n}\binom{n}{r}P_k^{n-r}(x)P_{k-1}^r(x)Q_{n-r+j}(x). \tag{7.16}$$

Special cases of (7.15) and (7.16) occurring when $k = 2$ are given in (7.3) and (7.5), respectively, in equivalent forms.

From (7.11) we deduce

$$P_3(x)P = P^3 - P_2(x)I, \tag{7.17}$$

whence

$$P_3^n(x)P^{n+j} = (P^3 - P_2(x)I)^n P^j, \tag{7.18}$$

from which it follows that

$$P_3^n(x)P_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{3(n-r)+j}(x)P_2^r(x). \tag{7.19}$$

Similarly,

$$P_3^n(x)Q_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{3(n-r)+j}(x)P_2^r(x). \tag{7.20}$$

More generally,

$$P_k^n(x)P_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} P_{k(n-r)+j}(x)P_{k-1}^r(x) \tag{7.21}$$

and

$$P_k^n(x)Q_{n+j}(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} Q_{k(n-r)+j}(x)P_{k-1}^r(x). \tag{7.22}$$

By (1.8) and (1.15), we may prove

$$P^{mr+k} = Q_m(x)P^{m(r-1)k} - (-1)^m P^{m(r-2)+k}. \tag{7.23}$$

Hence

$$P^{(mr+k)n} = P^{\{m(r-2)+k\}n}(Q_m(x)P^m - (-1)^m I)^n. \tag{7.24}$$

Equating appropriate elements yields

$$P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{i(m+1)} \binom{n}{i} P_{\{m(r-1)+k\}n-mi}(x)Q_m^{n-i}(x). \tag{7.25}$$

Putting $k = 0$ in (7.25) produces a formula for $P_{mrn}(x)$.

Again using (1.15), three times now, we obtain another form of (7.23):

$$P^{mr+k} = Q_{2m}(x)P^{m(r-2)+k} - P^{m(r-4)+k}. \tag{7.26}$$

Following the reasoning outlined in (7.24) and (7.25), we derive alternative formulas for $P_{(mr+k)n}(x)$ and $P_{mnr}(x)$ which closely resembly (7.24) and (7.25).

Equation (7.25) may be generalized further by extension of (7.26) to get

$$P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{i(ms+1)} \binom{n}{i} Q_{sm}^{n-i}(x)P_{\{m(r-s)+k\}n-msi}(x) \tag{7.27}$$

with a corresponding simplification for $P_{mrn}(x)$ when $k = 0$.

Since, by (7.23),

$$Q_m(x)P^{mr+k} = P^{m(r-1)+k}(P^{2m} + (-1)^m I) \tag{7.28}$$

we may demonstrate that

$$Q_m^n(x)P_{(mr+k)n}(x) = \sum_{i=0}^{n} (-1)^{mi} \binom{n}{i} P_{\{m(r+1)+k\}n-2mi}(x) \tag{7.29}$$

with a specialization when $k = 0$.

Arguments similar to those used to obtain the general result (7.27) may be utilized to prove that

$$Q_{ms}^{n}(x)P_{(mr+k)n} = \sum_{i=0}^{n}(-1)^{msi}\binom{n}{i}P_{\{m(r+s)+k\}n-2msi}(x) \tag{7.30}$$

leading to the simpler form when $k = 0$.

8. THE MATRIX SEQUENCE $\{n^V\}$

Ideas introduced in [5] for Fibonacci numbers are here expanded to apply to Pell polynomials.

Now, a generalization of the matrix P is the matrix

$$S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4x \\ 1 & 2x & 4x^2 \end{bmatrix}. \tag{8.1}$$

Induction demonstrates that

$$S^n = \begin{bmatrix} P_{n-1}^2(x) & P_{n-1}(x)P_n(x) & P_n^2(x) \\ 2P_{n-1}(x)P_n(x) & P_n^2(x) + P_{n-1}(x)P_{n+1}(x) & 2P_{n+1}(x)P_n(x) \\ P_n^2(x) & P_n(x)P_{n+1}(x) & P_{n+1}^2(x) \end{bmatrix} \tag{8.2}$$

The characteristic equation of S is

$$\lambda^3 - (4x^2+1)\lambda^2 - (4x^2+1)\lambda + 1 = 0. \tag{8.3}$$

From the Cayley-Hamilton theorem applied to (8.3), we have the recursion formula

$$S^n[S^3 - (4x^2+1)S(S+I) + I] = 0. \tag{8.4}$$

Corresponding elements in S^{n+3}, S^{n+2}, S^{n+1}, and S^n must satisfy (8.4). Therefore, from (8.2), we have the identities

$$P_{n+3}^2(x) - (4x^2+1)P_{n+2}^2(x) - (4x^2+1)P_{n+1}^2(x) + P_n^2(x) = 0 \tag{8.5}$$

and

$$P_{n+3}(x)P_{n+4}(x) - (4x^2+1)P_{n+2}(x)P_{n+3}(x) - (4x^2+1)P_{n+1}(x)P_{n+2}(x) + P_n(x)P_{n+1}(x) = 0. \tag{8.6}$$

[Parenthetically, we remark that the Cayley-Hamilton theorem may be employed with S to derive the sums given in (5.1) and (5.3).]

Again, after a little algebraic manipulation, the Cayley-Hamilton theorem leads to

$$(S+I)^3 = 4(x^2+1)S(S+I). \tag{8.7}$$

Mathematical induction establishes

$$(S+I)^{2n+1} = 4^n(x^2+1)^nS^n(S+I). \tag{8.8}$$

Now multiply both sides of (8.8) by S^j.

Equate corresponding elements to obtain

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} P_{k+j}^2(x) = 4^n(x^2+1)^n P_{2n+1+2j}(x) \quad \text{by [7, (3.20)].} \tag{8.9}$$

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} P_k(x)P_{k+j+1}(x) = 4^n(x^2+1)^n P_{2n+2+2j}(x). \tag{8.10}$$

By [7, (2.8)] we have

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} Q_k^2(x) = 4^{n+1}(x^2+1)^{n+1} P_{2n+1}(x),$$

while by (1.17) we have

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} Q_k(x)Q_{k+1}(x) = 4^{n+1}(x^2+1)^{n+1} P_{2n+2}(x)$$

with similar results to those in (8.9) and (8.10) when k is replaced by $k+j$ in (8.11) and (8.12).

If, now, in (8.8) we multiply both sides by $(S+I)S^j$, we get

$$\sum_{k=0}^{2n+2} \binom{2n+2}{k} P_{k+j}^2(x) = 4^n(x^2+1)^n Q_{2n+2j+2}(x) \tag{8.13}$$

and

$$\sum_{k=0}^{2n+2} \binom{2n+2}{k} P_{k+j}(x)P_{k+j+1}(x) = 4^n(x^2+1)^n Q_{2n+2j+3}(x). \tag{8.14}$$

When use is made of [7, (2.8)], (1.17), and both sides of the formula for (8.8) multiplied by $(S+I)S^j$, we derive

$$\sum_{k=0}^{2n+2} \binom{2n+2}{k} Q_{k+j}^2(x) = 4^{n+1}(x^2+1)^{n+1} Q_{2n+2j+2}(x) \tag{8.15}$$

and

$$\sum_{k=0}^{2n+2} \binom{2n+2}{k} Q_{k+j}(x)Q_{k+j+1}(x) = 4^{n+1}(x^2+1)^{n+1} P_{2n+2j+3}(x). \tag{8.16}$$

Extending the forms of the matrices P and S further, we have

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 6x \\ 0 & 1 & 4x & 12x^2 \\ 1 & 2x & 4x^2 & 8x^3 \end{bmatrix}$$

for which the characteristic equation is

$$\lambda^4 - (8x^3+4x)\lambda^3 - (16x^4+12x^2+2)\lambda^2 + (8x^3+4x)\lambda + I = 0, \tag{8.18}$$

From which are obtained (see [9]) forms for T^n and formulas for three cubic expressions in Pell polynomials corresponding to the two quadratic ones in (8.5) and (8.6), and an expression for

$$\sum_{r=1}^{n} P_r^3(x)$$

which is a variation of (4.16).

Matrices S and T are elements in a sequence of matrices $\{_nV\}$,

$$_1V = [1], \quad _2V = \begin{bmatrix} 0 & 1 \\ 1 & 2x \end{bmatrix}, \quad _3V = S, \quad _4V = T, \ \ldots, \ _rV, \ \ldots, \tag{8.19}$$

the order of $_rV$ being r.

The element $_r\nu_{ij}$ of $_rV$ in the i^{th} row and j^{th} column is

$$_r\nu_{ij} = \binom{j - 1}{j + i - r - 1}(2x)^{i+j-r-1}. \tag{8.20}$$

It is *conjectured* that the characteristic equation of $_rV$ is

$$\sum_{k=0}^{r} (-1)^{[k(k+1)]/2}\{r, k\}\lambda^{r-k} = 0, \tag{8.21}$$

where

$$\{r, k\} = \prod_{i=1}^{r}\{P_i(x)\} \Big/ \prod_{i=0}^{k}\{P_i(x)\}\prod_{i=1}^{r-k}\{P_i(x)\}, \quad 0 \leqslant k \leqslant r, \tag{8.22}$$

using the notation (extended) of [4]. That is, the symbol $\{r, k\}$ represents a generalization of a binomial coefficient. Following the ideas in [4], we note the results:

$$\{r, k\} = \{r, r - k\} \quad \text{by (8.22);} \tag{8.23}$$

$$\{r, r\} = 1 \quad \text{by (8.22);} \tag{8.24}$$

$$\{r, 0\} = 1 \quad \text{by (8.23) and (8.24);} \tag{8.25}$$

$$\{r, 1\} = \{r, r - 1\} = P_r(x) \quad \text{by (8.22) and (8.23).} \tag{8.26}$$

Next, we write

$$\{r, k\} = P_r(x)C(x), \tag{8.27}$$

whence

$$\{r - 1, k\} = P_{r-k}(x)C(x) \tag{8.28}$$

and

$$\{r - 1, k - 1\} = P_k(x)C(x). \tag{8.29}$$

Further,

$$\begin{aligned}\{r, k\} &= P_{r-k+k}(x)C(x) \\ &= P_{r-k}(x)P_{k+1}(x)C(x) + P_{r-k-1}(x)P_k(x)C(x) \quad \text{by [7, (2.14)],}\end{aligned}$$

so, by (8.28) and (8.29),

$$\{r, k\} = P_{r-k+1}(x)\{r - 1, k - 1\} + P_{k+1}(x)\{r - 1, k\}, \tag{8.30}$$

a type of Pascal triangle relationship.

Similarly,

$$\{r, k\} = P_{r-k-1}(x)\{r - 1, k - 1\} + P_{k-1}(x)\{r - 1, k\}. \tag{8.31}$$

Adding (8.30 and (8.31), and invoking [7, (3.24)], we deduce

$$2\{r,\ k\} = Q_{r-k}(x)\{r - 1,\ k - 1\} + Q_k(x)\{r - 1,\ k\}. \tag{8.32}$$

Going back to conjecture (8.21), we note that the expression for the symbol $\{r,\ k\}$ in (8.21) and (8.22) involves divisibility properties of the Pell polynomials. Although these are not discussed here, they are investigated in some detail in [9]. A key divisibility result proved in [9], for instance, is

$$P_m(x)\,|\,P_n(x) \quad \text{if and only if} \quad m\,|\,n. \tag{8.33}$$

The polynomial expressions occurring as powers of λ in (8.3) and (8.18), e.g., are $\{3,\ 1\}$ and $\{3,\ 2\}$, and $\{4,\ 1\}$, $\{4,\ 2\}$, and $\{4,\ 3\} = \{4,\ 1\}$, respectively.

9. CONCLUDING REMARKS

Naturally the consequences of the use of matrix methods in developing combinatorial number-theoretic properties of Pell and Pell-Lucas polynomials are by no means exhausted in our brief account above.

Quite apart from pursuing the discovery of additional formulas by the matrix techniques indicated, we can introduce different matrices to obtain new results.

Another interesting set of problems is to derive the sum of series whose terms are fractional and involve products of Pell or Pell-Lucas polynomials in the denominator, e.g.,

$$\sum_{r=1}^{n} \frac{(-1)^r}{P_r(x)P_{r+1}(x)}.$$

Putting $x = 1$ in the expression and summing to infinity, we may deduce the infinite alternating series summation involving Pell numbers,

$$\sum_{r=1}^{\infty} \frac{(-1)^r}{P_r P_{r+1}} = 1 - \sqrt{2}, \tag{9.1}$$

but enough has been said on our general theme for the moment.

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LETTER TO THE EDITOR

July 1, 1986

Over the years, several articles have appeared in *The Fibonacci Quarterly* relating the Fibonacci numbers to growth patterns in plants. Recently, Roger V. Jean, Professor of Mathematics and research worker in biomathematics at the University of Quebec has written the book *Mathematical Approach to Pattern and Form in Plant Growth* (Wiley & Sons), which should interest many readers of the *Quarterly*.

Dr. Jean addresses the mathematical problems raised by phyllotaxis, the study of relative arrangements of similar parts of plants and of technical concepts related to plant growth. He includes not only recent mathematical developments but also those that have appeared in specialized periodicals since 1830, listing well over 400 references. The book is written as a textbook for an advanced course in plant biology and mathematics or as a reference for workers in biomathematics. Besides that, it is just plain interesting reading.

Sincerely,

Marjorie Bicknell-Johnson