TENTH ROOTS AND THE GOLDEN RATIO

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1. INTRODUCTION

The ratio of the radius of a circle and a side of the inscribed regular decagon equals the golden ratio τ. In the complex plane the spines of a regular decagon inscribed in a circle of unit radius are the vector representations of the complex tenth roots of -1, if the decagon is appropriately turned. These two observations motivate an interest in expressing the tenth roots of -1 in terms of the golden ratio. The roots themselves may be derived using either the polar representation of -1, for it is known that they are expressible as $e^{\pi i r/10}$ when r is an integer, or they may be obtained algebraically, since when 5 divides n, the field of the n^{th} roots of -1 contains $\sqrt{5}$ and hence contains τ.

2. RÉSUMÉ ON THE GOLDEN RATIO

The golden ratio is the limiting ratio of two successive Fibonacci numbers. This limiting ratio satisfies the quadratic equation,

$$\tau^2 - \tau - 1 = 0 \tag{1}$$

in which the first root

$$\tau = \frac{1 + \sqrt{5}}{2} \tag{2}$$

is the golden ratio and the second root is

$$\frac{1 - \sqrt{5}}{2} = -\frac{1}{\tau}; \tag{3}$$

see [1].

The idea is to introduce the quantities (2) and (3) into the expressions calculated below for the tenth roots of -1.

3. THE POLAR APPROACH TO THE TENTH ROOTS OF -1

Since -1 has unit modulus and an argument of 180°, its polar representation is

$$-1 = \cos\theta_n + i \sin\theta_n \tag{4}$$

with

$$\theta_n = 180° + 360°n, \tag{5}$$

where $n = 0, \pm1, \pm2, \ldots$ accounts for the periodicity of circular measure, and $i^2 = -1$. The tenth roots of -1 are then given by

$$\cos\frac{\Theta_n}{10} + i \sin\frac{\Theta_n}{10}$$

which in complex rectangular form are, successively,

$$\begin{aligned} Z_{0,1} &= i, \\ Z_{2,3,4,5} &= \pm(\cos 18° \pm i \sin 18°), \\ Z_{6,7,8,9} &= \pm(\cos 54° \pm i \sin 54°), \end{aligned} \tag{6}$$

each subscript denoting a different choice of algebraic sign.

The golden ratio is introduced by expressing the trigonometric ratios in (6) as surds. To do this, first use the result

$$\sin(2 \times 18°) = \sin 36° = \cos(90° - 36°) = \cos 54° = \cos(3 \times 18°) \tag{7}$$

to obtain, with the respective double and triple angle formulas for the sine and cosine,

$$2 \sin 18° \cos 18° = 4 \cos^3 18° - 3 \cos 18°. \tag{8}$$

Next, divide both sides of (8) by cos 18° to reduce it to a quadratic equation in sin 18°, viz,

$$4 \sin^2 18° + 2 \sin 18° - 1 = 0, \tag{9}$$

with positive root

$$\sin 18° = \frac{-1 + \sqrt{5}}{4} = \frac{1}{2\tau}. \tag{10}$$

Then we can write

$$\begin{aligned} \cos 18° &= \sqrt{1 - \sin^2 18°} = \sqrt{1 - \left(\frac{1}{2\tau}\right)^2} = \frac{\sqrt{3 + 4\tau}}{2\tau} \\ &= \frac{(1 + \tau)\sqrt{3 - \tau}}{2\tau} = \frac{\tau\sqrt{3 - \tau}}{2}, \end{aligned} \tag{11}$$

where equation (1) has also been used. Furthermore,

$$\cos 54° = \sin 36° = 2 \sin 18° \cos 18° = \frac{\sqrt{3 - \tau}}{2}$$

and

$$\sin 54° = \sqrt{1 - \cos^2 54°} = \sqrt{1 - \left(\frac{3 - \tau}{4}\right)} = \frac{\sqrt{1 + \tau}}{2} = \frac{\tau}{2}. \tag{12}$$

According to these expressions, the tenth roots of -1 become

$$\begin{aligned} Z_{0,1} &= \pm i, \\ Z_{2,3,4,5} &= \pm\frac{1}{2}\left(\tau\sqrt{3 - \tau} \pm i\,\frac{1}{\tau}\right). \\ Z_{6,7,8,9} &= \pm\frac{1}{2}(\sqrt{3 - \tau} + i\,\tau), \end{aligned} \tag{13}$$

and they may be sketched in the Argand plane as in Figure 1.

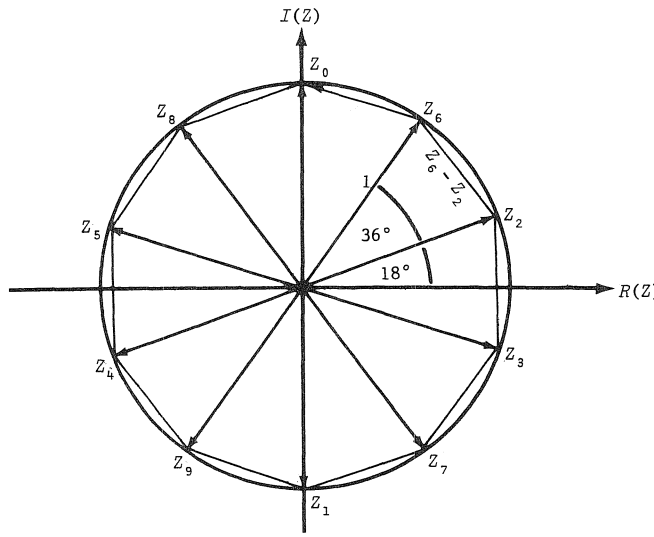

Figure 1

Since the side of the corresponding decagon is the modulus of the difference of two successive roots, we see from the figure that the ratio alluded to in the Introduction is typically

$$1/|Z_6 - Z_2| = \frac{1}{2 \sin 18°} = \tau \tag{14}$$

from (10).

4. ALGEBRAIC APPROACH

The tenth roots of -1 satisfy

$$Z^{10} + 1 = 0 \tag{15}$$

or, replacing Z^2 by ξ, say,

$$\xi^5 + 1 = 0. \tag{16}$$

This shows that the golden ratio is also relevant to an investigation of the fifth roots of -1. The golden ratio arises, for instance, in the geometry of the regular five-pointed star.

Equation (16) can be factorized to

$$(\xi + 1)(\xi^4 - \xi^3 + \xi^2 - \xi + 1) = 0, \tag{17}$$

showing that $\xi = -1$ is a root of the quintic in (16) and confirming that

$$Z = \pm\sqrt{-1} = \pm i \tag{18}$$

are two roots of the corresponding "dectic" in (15).

Dividing the remaining quartic factor in (17) by ξ^2 gives

$$\xi^2 + \frac{1}{\xi^2} - \left(\xi + \frac{1}{\xi}\right) + 1 = 0, \tag{19}$$

which on substituting

$$\eta = \xi + \frac{1}{\xi} \tag{20}$$

reduces to

$$\eta^2 - \eta - 1 = 0, \tag{21}$$

with roots as in (1), viz,

$$\eta_{1,2} = \tau, \; -\frac{1}{\tau}. \tag{22}$$

From (20), we also have

$$\xi^2 - \eta\xi + 1 = 0 \tag{23}$$

with roots

$$\xi = \frac{\eta \pm \sqrt{\eta^2 - 4}}{2} = \frac{\eta \pm \sqrt{\eta - 3}}{2}. \tag{24}$$

Inserting the appropriate values of η given in (22), we obtain the complex fifth roots of -1 as

$$\xi_{1,2} = \frac{1}{2}(\tau \pm i\sqrt{3 - \tau})$$

and

$$\xi_{3,4} = \frac{1}{2}\left(-\frac{1}{\tau} \pm i\sqrt{3 - \tau}\right), \tag{25}$$

from which required complex tenth roots follow with, for instance,

$$Z = \pm\sqrt{\xi}. \tag{26}$$

These square roots are found by proceeding typically as follows. Let

$$a + jb = \sqrt{\frac{1}{2}(\tau + j\sqrt{3 - \tau})}. \tag{27}$$

Since the right-hand side is a root of -1, we have

$$a^2 + b^2 = 1. \tag{28}$$

Also, squaring both sides in (27) and equating real and imaginary parts in the result, we arrive at, with a little help from (1),

$$a^2 - b^2 = \frac{\tau}{2} \tag{29}$$

and

$$ab = \frac{\sqrt{3 - \tau}}{2}. \tag{30}$$

These indicate that the product of a and b is positive, meaning that a and b are together either both positive or both negative. Solving (28) and (29) simultaneously gives

$$a^2 = \frac{1}{2}\left(1 + \frac{\tau}{2}\right) = \frac{2 + \tau}{4} = \frac{(2 + \tau)\tau^2}{4\tau^2} = \frac{4\tau + 3}{4\tau^2} = \frac{[\tau^2(3 - \tau)]}{4},$$

from which

$$a = \pm \frac{\tau\sqrt{3 - \tau}}{2}$$

and

$$b^2 = \frac{1}{2}\left(1 - \frac{\tau}{2}\right) = \frac{2 - \tau}{4} - \frac{(2 - \tau)}{4\tau^2}\,\tau^2 = \frac{(2 - \tau)(1 + \tau)}{4\tau^2} = \frac{1}{4\tau^2},$$

$$\therefore\ b = \pm \frac{1}{2\tau}.$$

Thus, from the square root in (27), we obtain two of the tenth roots in (13), namely,

$$\sqrt{\frac{1}{2}(\tau + j\sqrt{3 - \tau})} = \pm\frac{1}{2}\left(\tau\sqrt{3 - \tau} + j\left(\frac{1}{\tau}\right)\right). \tag{31}$$

The other tenth roots in (13) can be obtained similarly from the fifth roots in (25).

Of course, the same procedure outlined here is applicable to the problem of expressing the fifth and tenth roots of unity (i.e., +1 rather than -1), in terms of the golden ratio; however, this is left as an exercise for the interested reader.

REFERENCE

1. Verner E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Boston: Houghton Mifflin, 1969.

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