DIFFERENCES BETWEEN SQUARES AND POWERFUL NUMBERS

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A number P is *powerful* if, whenever a prime p divides P, then p also divides P. In [2] McDaniel proves that each nonzero integer can be written in infinitely many ways as the difference between two relatively prime powerful numbers. (Golomb [1] had conjectured that infinitely many integers could not be represented as the difference between powerful numbers.) An examination of McDaniel's paper shows that he actually proves that, if $n \not\equiv 2 \pmod 4$, then n can be written in infinitely many ways as $S - P$, where S is a square, P is powerful, and $(S, P) = 1$.

In this paper we take care of the case $n \equiv 2 \pmod 4$, to prove

Theorem: If n is any nonzero integer, then n can be written in infinitely many ways as $n = S - P$, where S is a square, P is powerful, and $(S, P) = 1$.

Proof: For compactness, we assume the reader is familiar with [2]. In Theorem 2 of that paper it is proved that if n is a positive integer and $n \not\equiv 2 \pmod 4$ then $x^2 - Dy^2 = n$ has infinitely many relatively prime solutions X, Y such that D divides Y. Clearly, each represents n in the desired way. The method of proof is to show that there exist integers D, p, q, x_0, and y_0 such that

$$D > 0 \text{ and } D \text{ is not a square,} \tag{1}$$

$$p \text{ and } q \text{ satisfy } p^2 - Dq^2 = n \text{ and } (p, q) = 1, \tag{2}$$

$$x_0 \text{ and } y_0 \text{ satisfy } x^2 - Dy^2 = \pm 1, \tag{3}$$

$$(2py_0, D) \text{ divides } q. \tag{4}$$

Although McDaniel assumes $n > 0$ in the proof of his Theorem 2, the arguments he gives work just as well for negative values of n. Thus, only the case $n \equiv 2 \pmod 4$ remains. Let $n = 8k \pm 2$.

Case 1. $n = 8k + 2$ or $3 \nmid n$.

If $n = 2$, then $D = 7$, $p = 3$, $q = 1$, $x_0 = 8$, and $y_0 = 3$ can be checked to satisfy (1) through (4). Likewise, if $n = 10$, then $D = 39$, $p = 7$, $q = 1$, $x_0 = 25$, and $y_0 = 4$ work.

Otherwise, we take $D = (2k - 1)^2 \mp 2$, $p = 2k + 1$, $q = 1$, $x_0 = D \pm 1$, and $y_0 = 2k - 1$. Since $n = 2$ and $n = 10$ have been excluded, we see that $D > 1$ and D is odd. Conditions (2) and (3) are easily checked. Note that because $p^2 - D = n$, we have $p^2 - D - 4p = \pm 2 - 4 = -2$ or -6. Since D is odd, $(p, D) = 1$ or 3, with the latter a possibility only if we take the bottom signs. However, $(p, D) = 3$ implies $3 | n$, contrary to our assumption. Thus, $(p, D) = 1$. Also, $y_0^2 - D = \pm 2$, so $(y_0, D) = 1$. This proves (4).

Case 2: $n = 8k - 2$ and $3 | n$.

We take $p = 6k - 1$, $q = 1$, $D = p^2 - n = 36k^2 - 20k + 3$, $x_0 = 9D - 1$, and $y_0 = 3(18k - 5)$. It can be checked that $D > 1$ and that D is strictly between

p^2 and $(p - 1)^2$ for any value of k. We calculate that $y_0^2 - 81D = -18$, and so $x_0^2 - Dy_0^2 = (9D - 1)^2 - D(81D - 18) = 1$, while (2) is immediate. Note that $3 \nmid p$ but $3|n$, so $3 \nmid D$. Since D is odd, we see that $(y_0, D) = 1$. Finally,

$$3(p^2 - D) - 4p = 3n - 4p = -2,$$

and so $(p, D) = 1$ also.

To compute solutions to $S - P = n$, we can follow McDaniel and define integers x_j, y_j for $j > 0$ by

$$x_j + y_j\sqrt{D} = (x_0 + y_0\sqrt{D})^{cj},$$

where $c = 2$, then take

$$S = (px_j + Dqy_j)^2 \quad \text{and} \quad P = D(py_j + qx_j)^2,$$

where j is any positive solution to $(cpy_0)j \equiv -qx_0 \pmod{D}$. If $x_0^2 - Dy_0^2 = +1$, however, such as in the present case and in McDaniel's treatment of the case $n = 4k + 1$, sometimes a smaller solution may be found by taking $c = 1$ in the above discussion. This gives a smaller solution when the least positive solution to $(py_0)j \equiv -qx_0 \pmod{D}$ is less than twice the least positive solution to $(2py_0)j \equiv -qx_0 \pmod{D}$, and, in any case (when $x_0^2 - Dy_0^2 = 1$), more solutions are obtained this way. If $n = 14$, for example, we generate solutions

$$S = (5x + 11y)^2 \quad \text{and} \quad P = 11(5y + x)^2,$$

where x and y are defined so that

$$x + y\sqrt{11} = (10 + 3\sqrt{11})^{3+11t} \text{ or } (10 + 3\sqrt{11})^{2(7+11t)}, \; t \geqslant 0,$$

depending on whether we take $c = 1$ or 2.

It has been proved by McDaniel [3] and Mollin and Walsh [4, 5] that every nonzero integer can be written in infinitely many ways as the difference of two relatively prime powerful numbers, *neither* of which is a square.

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