A FAMILY OF FIBONACCI-LIKE SEQUENCES

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We consider the recurrence relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j,$$

where $G_0 = G_1 = 1$, and we express G_n in terms of the Fibonacci numbers F_n and F_{n-1}, and in the parameters $\alpha_0, \ldots, \alpha_k$.

For integer values of $k, \alpha_0, \ldots, \alpha_k$, the relation

$$G_n = G_{n-1} + G_{n-2} + \sum_{j=0}^{k} \alpha_j n^j, \tag{1}$$

where $G_0 = G_1 = 1$, forms a difference equation that can be solved by standard methods. In this note, we provide such a solution for equations of this type, in which we treat $\alpha_0, \ldots, \alpha_k$ as parameters.

First, the solution $G_n^{(h)}$ of the corresponding homogeneous equation equals

$$G_n^{(h)} = C_1\phi_1^n + C_2\phi_2^n,$$

where $\phi_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\phi_2 = \frac{1}{2}(1 - \sqrt{5})$; cf. e.g., [1] and [3].

Second, as a particular solution, we try

$$G_n^{(p)} = \sum_{i=0}^{k} A_i n^i,$$

which yields

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k} A_i (n-1)^i - \sum_{i=0}^{k} A_i (n-2)^i - \sum_{i=0}^{k} \alpha_i n^i = 0$$

or

$$\sum_{i=0}^{k} A_i n^i - \sum_{i=0}^{k}\left(\sum_{\ell=0}^{i} A_i \binom{i}{\ell}(-1)^{i-\ell}(1 + 2^{i-\ell})n^\ell\right) - \sum_{i=0}^{k} \alpha_i n^i = 0.$$

For each i $(0 \leqslant i \leqslant k)$, we have

$$A_i - \sum_{m=i}^{k} \beta_{im} A_m - \alpha_i = 0, \tag{2}$$

where, for $m \geqslant i$,

$$\beta_{im} = \binom{m}{i}(-1)^{m-i}(1 + 2^{m-i}).$$

From the recurrence relation (2), $A_k, \ldots, A_0$ can be computed (in that order): A_i is a linear combination of $\alpha_i, \ldots, \alpha_k$. However, a more explicit expression for A_i can be obtained by setting

$$A_i = -\sum_{j=i}^{k} a_{ij}\alpha_j.$$

(The minus sign happens to be convenient in the sequel.) Then (2) implies

$$-\sum_{j=i}^{k} a_{ij}\alpha_j + \sum_{m=i}^{k} \beta_{im}\left(\sum_{\ell=m}^{k} a_{m\ell}\alpha_\ell\right) - \alpha_i = 0.$$

Since $\beta_{ii} = 2$, we have, for $0 \leqslant i \leqslant k$,

$$a_{ii} = 1$$

$$a_{ij} = -\sum_{m=i+1}^{j} \beta_{im}a_{mj}, \text{ if } j > i.$$

Hence,

$$G_n^{(P)} = -\sum_{i=0}^{k}\sum_{j=i}^{k} a_{ij}\alpha_j n^i = -\sum_{j=0}^{k}\alpha_j\left(\sum_{i=0}^{j} a_{ij}n^i\right).$$

Finally, we ought to determine C_1 and C_2: $G_0 = G_1 = 1$ implies

$$C_1 + C_2 = 1 - G_0^{(p)},\quad C_1\phi_1 + C_2\phi_2 = 1 - G_1^{(p)}.$$

These equalities yield

$$\begin{aligned} C_1 &= ((G_0^{(p)} - 1)\phi_2 + 1 - G_1^{(p)})(\sqrt{5})^{-1} \\ &= ((1 - G_0^{(p)})\phi_1 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1}, \end{aligned}$$

$$\begin{aligned} C_2 &= ((G_0^{(p)} - 1)\phi_1 + G_1^{(p)} - 1)(\sqrt{5})^{-1} \\ &= -((1 - G_0^{(p)})\phi_2 - G_1^{(p)} + G_0^{(p)})(\sqrt{5})^{-1}, \end{aligned}$$

and

$$G_n = (1 - G_0^{(p)})F_n + (-G_1^{(p)} + G_0^{(p)})F_{n-1} + G_n^{(p)}.$$

Summarizing, we have the following proposition.

Proposition: The solution of (1) can be expressed as

$$G_n = (1 + \Lambda_k)F_n + \lambda_k F_{n-1} - \sum_{j=0}^{k}\alpha_j p_j(n),$$

where Λ_k is a linear combination of $\alpha_0, \ldots, \alpha_k$, λ_k is a linear combination of $\alpha_1, \ldots, \alpha_k$, and for each j $(0 \leqslant j \leqslant k)$, $p_j(n)$ is a polynomial of degree j:

$$\Lambda_k = \sum_{j=0}^{k} a_{0j}\alpha_j,\quad \lambda_k = \sum_{j=1}^{k}\left(\sum_{i=1}^{j} a_{ij}\right)\alpha_j,\quad p_j(n) = \sum_{i=0}^{j} a_{ij}n^i.$$

Remarks:

(1) For $j = 0, 1, \ldots, 8$, the polynomials $p_j(n)$ are given in Table 1.

(2) No assumptions on $\alpha_0, \ldots, \alpha_k$ have been made; thus, they may be rational or real numbers as well.

(3) Changing $G_1 = 1$ into $G_1 = c$ only affects λ_k; it has to be increased with $c - 1$.

Table 1

j	$p_j(n)$
0	1
1	$n + 3$
2	$n^2 + 6n + 13$
3	$n^3 + 9n^2 + 39n + 81$
4	$n^4 + 12n^3 + 78n^2 + 324n + 673$
5	$n^5 + 15n^4 + 130n^3 + 810n^2 + 3365n + 6993$
6	$n^6 + 18n^5 + 195n^4 + 1620n^3 + 10095n^2 + 41958n + 87193$
7	$n^7 + 21n^6 + 273n^5 + 2835n^4 + 23555n^3 + 146853n^2 + 610351n + 1268361$
8	$n^8 + 24n^7 + 364n^6 + 4536n^5 + 47110n^4 + 391608n^3 + 2441404n^2 + 10146888n + 21086113$

(4) The coefficients of $\alpha_0, \alpha_1, \alpha_2, \ldots$ in Λ_k and of $\alpha_1, \alpha_2, \ldots$ in λ_k are independent of k. Thus, they give rise to two infinite sequences Λ and λ of natural numbers, as k tends to infinity, of which the first few elements are

Λ: 1, 3, 13, 81, 673, 6993, 87193, 1268361, 21086113, ...,

λ: 1, 7, 49, 415, 4321, 53887, 783889, 13031935, ...

Neither of these sequences is included in [2].

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