INCREDIBLE IDENTITIES REVISITED

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Consider the numbers:

$$A = \sqrt{2 \cdot 11 + 2\sqrt{5}} + \sqrt{5};$$

$$B = \sqrt{11 + \sqrt{116}} + \sqrt{(11 + 5) - \sqrt{116} + 2\sqrt{5(11 - \sqrt{116})}}.$$

Although one feels that these numbers couldn't be equal, Shanks [2] assures us that they are. Indeed, Follin (as reported by Spohn [3]) points out that one may take 5, 11, and 116 as *indeterminates* subject only to the identity

$$5 = 11^2 - 116 \tag{1}$$

(which certainly is true for the usual interpretation of these strings of decimal digits). As we shall see, it is only the first 5 in A which needs to be given by the representation (1); the remaining 5's may be treated as a separate indeterminate. The proofs of the equality of A and B given in [2] and [3] seem to be little more than appeals to the principle, attributed to J. Littlewood, that "any identity, once written down, is trivial."

Please ask yourself the following questions before reading further:

1. Why does $A = B$ seem so unlikely?
2. Given that it is true that $A = B$, how can it be proved?

The answers to both questions can be traced to the same source, Book X of Euclid's *Elements* [1]. Indeed, in Proposition 42, it is shown that a number expressible as a sum of two incommensurate square roots of *rational* numbers has a unique such representation up to interchanging the order of the summands. This deals with question 1.

Much of Euclid's work deals with more complicated algebraic numbers, albeit only *constructible* numbers. In this analysis, repeated use is made of the rule

$$\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}, \tag{2}$$

which is employed forward and backward. That is, to take the square root of a quantity like $22 + 2\sqrt{5}$, one solves

$$\begin{aligned} a + b &= 22 \\ ab &= 5 \end{aligned} \tag{3}$$

to obtain a and b as $11 + \sqrt{116}$ and $11 - \sqrt{116}$. At this point, it is clear that our quantities A and B are the two different ways of associating

$$\sqrt{11 + \sqrt{116}} + \sqrt{11 - \sqrt{116}} + \sqrt{5}$$

using (2) to express the first sum that one takes in each case. Q.E.D.

Equation (2) has led to puzzles before. You can discover one by using the method (3) to obtain another expression for

$$\sqrt{2 + 2\sqrt{2}}.$$

One case where the method has a fairly satisfying answer is

$$\sqrt{5 + 2\sqrt{6}}.$$

Finally, while it seems that, in the case of

$$\sqrt{22 + 2\sqrt{5}},$$

the method has caused the complication to *ramify*, it does not lead to *proliferation*. To see this, find

$$\sqrt{11 + 2\sqrt{29}}.$$

Although Euclid's study of algebraic numbers is full of detailed discussion of points which seem to us to be misguided, it is sobering to note that it can lead to a natural explanation of an identity that is not very close to the surface in our modern theory of algebraic numbers.

REFERENCES

1. T. L. Heath. *The Thirteen Books of Euclid's Elements*, Vol. III. Cambridge: Cambridge University Press, 1908.
2. D. Shanks. "Incredible Identities." *The Fibonacci Quarterly* **12**, no. 3 (1974):271, 280.
3. W. G. Spohn, Jr. Letter to the Editor. *The Fibonacci Quarterly* **14**, no. 1 (1976):12.

AFTERTHOUGHTS

Since composing the article, I have corresponded with Professor Shanks and others whose interest in this topic came to light in that correspondence. It seems that everyone has his own favorite proof of this identity, usually reflecting the individual's background in classical algebra.

It also appears that different types of proofs have different gestation times. The proof in Spohn's letter had multiple independent discoveries at that time, and a proof along the lines of my article was communicated to Shanks by J. G. Wendel of the University of Michigan in October 1984.

In all proofs, two separate parts must be distinguished. First, the quantities A and B can be shown to satisfy the same polynomial with rational coefficients, i.e., to be algebraically conjugate. This is most susceptible to proof by Littlewood's principle. To show that the numbers are actually equal as real numbers relies on special knowledge of the real roots of that polynomial. This is hidden in my proof because I need only distinguish the two square roots of a real number. Another tool which is used in my proof (but could be overlooked) is the fact that the sum of algebraic numbers is algebraic.

Shanks also notes that his proof is really a means of discovery of such identities, and he refers the reader to his article [4].

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ADDITIONAL REFERENCE

4. D. Shanks. "A Survey of Quadratic, Cubic, and Quartic Algebraic Number Fields (From a Computational Point of View)." *Proc. 7th SE Conference on Combinatorics, Graph Theory and Computing*, 1976, pp. 15-40.

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