THE NAMING OF POPES AND A FIBONACCI SEQUENCE IN TWO NONCOMMUTING INDETERMINATES

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The naming of Popes is a serious matter,
It isn't just one of your holiday games.
I know you may think I'm as mad as a hatter,
But I say that a Pope must have two different names.

—Apologies to T. S. Eliot and
Old Possum's Book of Practical Cats

The year 1978 saw three occupations of the Chair of St. Peter and the second was the shortest reign of modern times. Luciano Albini was acclaimed as the successor to Paul VI but was fated to be Christ's Vicar on Earth for only a month. He nevertheless introduced a novelty. So impressed was he by his two predecessors that he chose the double appellation of John-Paul. The innovation seemed to meet general approval, as it affirmed continuity in Church policy while paying tribute to the two previous pontiffs. However, it was a dangerous precedent and it was fortunate indeed that the present Bishop of Rome did not feel obliged to follow his predecessor's example, but prudently opted simply to extend the line of John-Pauls. Indeed, a moment's reflection will reveal that if John-Paul I had insisted that all his successors should follow his lead in this matter the effect on papal nomenclature would have been catastrophic, although of considerable mathematical interest.

Disaster was averted, but let us look at the mathematics anyway. Suppose that John-Paul I had insisted that each future pope should take as his name the names of his two predecessors in chronological order. Commencing with Pope John XXIII, the "papal sequence," as we shall call it, would begin

$$J,\ P,\ JP,\ PJP,\ JP^2JP,\ PJPJP^2JP,\ JP^2JP^2JPJP^2JP,\ \ldots,$$

where J, P, and P^2 have their obvious meanings. An impossible situation for the popes of the third millenium; each would spend a great deal of time trying to remember his own name. However, this same sequence should delight the heart of any lover of the Golden Ratio because it can be regarded as a Fibonacci sequence in two noncommuting generators, J and P. We shall study this sequence with an eye to finding an efficient algorithm to determine P_n, the name of the n^{th} pope, where we shall take P_1 to be Pope John-Paul himself.

We shall begin with several simple observations. Denote the length of P_n by $|P_n|$, and denote by $|P_n|_J$ and $|P_n|_P$ the number of occurrences of John and Paul, respectively, in P_n. We use F_n to denote the n^{th} Fibonacci number.

Lemma 1: In the papal sequence, for all $n \geqslant 1$,

(i) $|P_n|_J = F_n, \quad |P_n|_P = F_{n+1}$;

(ii) $|P_n| = F_{n+2}$;

(iii) P_m ends in P_n for all $m \geqslant n$;

(iv) P_n does not contain two successive J's nor three successive P's.

Proof: Each of (i), (ii), and (iii) follow immediately from the definition of the papal sequence and induction.

(iv). From (iii) with $n = 1$, it follows that P_m ends in JP for all $m \geqslant 1$. It is obvious, then, that J^2 can never occur in the papal sequence. Next, obseve that P_n begins with PJ or with JP, according as n is even or odd (again this is immediate by induction). Hence, no P_n begins nor ends in P^2, a fact that ensures that P^3 never appears in our sequence.

This lemma allows us to reformulate our problem. Denote the reverse of P_n by $\bar{P}_n$. We associate with the papal sequence an infinite sequence $A = (a_i)_{i \in N}$, in which each a_i is either J, P, or P^2, by defining a_n to be the n^{th} term in $\bar{P}_m$ (read as a word in J, P, and P^2) for all m such that $|\bar{P}_m|$ is sufficiently long for this to make sense. Part (iii) of Lemma 1 guarantees that A is well-defined (to be precise, we should take m such that the length of $\bar{P}_m$, considered as a work in J, P, and P^2, is at least $n + 1$).

Since our problem is now of more mathematical than religious interest, we shall dispense with J, P, and P^2, replacing them by the symbols 0, 1, and 2, respectively. Since $|P_n|$ is known (up to the value of F_{n+2}), the papal sequence can be reconstructed from our sequence A. Furthermore, A begins in 1, and part (iv) of Lemma 1 tells us that A is a sequence in which each 1 and 2 is preceded and followed by 0, while 00 never occurs. Therefore, A can be reconstructed from the sequence B, which is obtained from A by deleting all the 0's (given that A begins in 1).

Our problem, then, is to discover a good way of generating this sequence B, which begins 12122..., the first five numbers corresponding to $\bar{P}_5$.

We introduce a sequence of finite sequences $B_0, B_1, B_2, \ldots$ (each of which, as we shall show, is an initial subsequence of its successor and of B). The sequence is defined recursively beginning $B_0 = 1$. We construct B_{m+1} from B_m by replacing each 1 by 12 and each 2 by 122. The next few B_i's are

B_1: 12,

B_2: 12122,

B_3: 1212212122122,

B_4: 1212212122122121221212212212122122.

Each B_i is an initial subsequence of its immediate (and hence of each) successor. Indeed, we can say more.

Lemma 2: For $n > 1$,

$$B_n = B^2_{n-1}B_{n-2}B_{n-3}\ldots B_1 2,$$

the product being concatenation of the sequences.

Proof: We denote by Q the operation defined in the recursive definition of $(B_i)_{i \in N^0}$, that is

$$Q(B_i) = B_{i+1},\ i = 0, 1, 2, \ldots .$$

The result is evidently true for $n = 2$. For $n \geqslant 3$, we obtain

$$\begin{aligned} B_n &= Q(B_{n-1}) = Q(B_{n-2}^2 B_{n-3} \ldots B_1 2) \text{ by the inductive hypothesis,} \\ &= Q(B_{n-2})Q(B_{n-2})Q(B_{n-3}) \ \ldots \ Q(B_1)Q(2) \\ &= B_{n-1}B_{n-1}B_{n-2} \ldots B_2 122 \\ &= B_{n-1}^2 B_{n-2} \ldots B_2 B_1 2. \end{aligned}$$

Remark: We can regard members of the sequence $(B_i)_{i \in N^0}$ as a set of generators for a semigroup S, whose multiplication is defined by concatenation. The operator $Q : S \to S$ is then seen to be an injective semigroup endomorphism.

Henceforth, we shall regard $\overline{P}_n$ as a finite sequence in 0, 1, 2, and, moreover, we shall agree to delete the 0's (as $\overline{P}_n$ can be recovered even if the 0's are deleted), but we shall denote this reduced version of $\overline{P}_n$ by the same symbol.

Lemma 3: For each $n \geqslant 1$, B_n is an initial subsequence of B. In fact,

$$B_n = \overline{P}_{2n+1}.$$

Proof: The proof is by induction. We shall prove the two identities

$$B_n = \overline{P}_{2n+1} \quad \text{and} \quad B_n B_{n-1} \ldots B_1 2 = \overline{P}_{2n} \cdot \overline{P}_{2n+1}, \ n \geqslant 1,$$

where the product on the right-hand side is defined by concatenation, with the understanding that two adjacent 1's are replaced by 2.

For $n = 1$, we have $B_1 = 12 = \overline{P}_3$ (as P_3 is JP^2JP), and $B_1 2 = \overline{P}_2 \cdot \overline{P}_3$, since $B_1 2 = 122$, while $\overline{P}_2 \cdot \overline{P}_3 = 11 \cdot 12 = 122$. Our inductive hypothesis is that

$$B_m = \overline{P}_{2m+1}$$

and

$$B_m B_{m-1} \ldots B_1 2 = \overline{P}_{2m} \cdot P_{2m+1} \text{ for all } 1 \leqslant m < n, \ n > 1.$$

Now, by Lemma 2, we have

$$B_n B_{n-1} \ldots B_1 2 = B_{n-1}^2 B_{n-2} \ldots B_1 2 B_{n-1} B_{n-2} \ldots B_1 2,$$

which, by the inductive hypothesis is equal to

$$\begin{aligned} &\overline{P}_{2n-1} \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) \\ &= (\overline{P}_{2n-1} \cdot \overline{P}_{2n-2}) \cdot (\overline{P}_{2n-1} \cdot \overline{P}_{2n-2}) \cdot \overline{P}_{2n-1} \\ &= \overline{P}_{2n} \cdot (\overline{P}_{2n} \cdot \overline{P}_{2n-1}) = \overline{P}_{2n} \cdot \overline{P}_{2n+1}. \end{aligned}$$

Hence, by Lemma 2,

$$\begin{aligned} B_n &= B_{n-1}^2 B_{n-2} \ldots B_1 2 \\ &= B_{n-1}(B_{n-1}B_{n-2} \ldots B_1 2) \\ &= \overline{P}_{2n-1} \cdot (\overline{P}_{2n-2} \cdot \overline{P}_{2n-1}) = (\overline{P}_{2n-1} \cdot \overline{P}_{2n-2}) \cdot \overline{P}_{2n-1} \\ &= \overline{P}_{2n} \cdot \overline{P}_{2n-1} = \overline{P}_{2n+1}, \end{aligned}$$

as required.

Result: Algorithm for constructing the papal sequence.

Odd Case: Suppose $n = 2m + 1$, $m \geqslant 0$.

1. Calculate $B_m = Q^m(1)$.
2. Write $\overline{B}_m$, the reverse of B_m.
3. Write 0 at the beginning and between each pair of symbols of $\overline{B}_m$.
4. Replace each 0, 1, and 2 by J, P, and P^2, respectively.

Even Case: Suppose $n = 2m$, $m \geqslant 1$.

1. Calculate $B_m = Q^m(1)$.
2. Suppose $B_m = b_1 b_2 \ldots b_k$, say. Truncate B_m at $B'_m = b_1 b_2 \ldots b_t$, where
$$\sum_{i=1}^{t} b_i = F_{2m+1} + 1.$$
Replace $b_t = 2$ by 1 in B'_m to give B''_m.
3. Write $\overline{B}''_m$.
4. Insert 0 between each pair of symbols of $\overline{B}''_m$.
5. Replace each 0, 1, and 2 by J, P, and P^2, respectively.

Proof: The algorithm for the odd case is an immediate consequence of Lemma 3 together with the observations made on the occurrences of J in P_n, when n is odd.

On the other hand, if $n = 2m$, then $B_m = \overline{P}_{2m+1}$. But P_{2m+1} ends in P_{2m}, and so some initial subsequence of B_m corresponds to $\overline{P}_{2m}$. The remaining problem is to determine the length of this subsequence. Now, by Lemma 1(i),

$$|P_{2m}|_P = F_{2m+1};$$

thus, we need to truncate $B_m = b_1 b_2 \ldots b_k$ at b_t, where t is the least integer such that

$$\sum_{i=1}^{t} b_i \geqslant F_{2m+1}.$$

Finally, observe that P_{2m} begins PJ, P_{2m-1} ends in JP, whence $b_t = 2$ and

$$\sum_{i=1}^{t} b_i = F_{2m+1} + 1.$$

The result follows from these observations.

Example: P_4. Here, $m = 2$.

1. $B_2 = Q^2(1) = Q(12) = 12122$.
2. $F_{2m+1} = F_5 = 5$, so $F_{2m+1} + 1 = 6$. Hence, $B'_2 = 1212$.
3. $\overline{B}''_2 = 1121$.

4. $1121 \to 1010201$.
5. P_4 is $PJPJP^2JP$.

Example: P_7. Here, $m = 3$.

1. $B_3 = Q^3(1) = Q^2(12) = Q(12122) = 1212212122122$.
2. $\overline{B}_3 = 2212212122121$.
3. $\overline{B}_3 \to 020201020201020102020102 01$.
4. P_7 is $JP^2JP^2JPJP^2JP^2JPJP^2JPJP^2JP^2JPJP^2JP$.