ON NONSQUARE POWERFUL NUMBERS

R. A. MOLLIN[1] and P. G. WALSH[2]
University of Calgary, Calgary, Alberta, Canada T2N 1N4
(Submitted March 1985)

1. INTRODUCTION

As introduced by Golomb in [1], a *powerful number* n is a positive integer which has no prime appearing to the first power in its canonical prime decomposition; i.e., if a prime p divides n, then p^2 divides n. If n and m are powerful numbers, then $n - m$ is said to be a *proper difference* of powerful numbers if g.c.d. $(n, m) = 1$. Golomb [1] conjectured that there are infinitely many integers which are not proper differences of powerful numbers. This was disproved by McDaniel in [3], wherein he gave an existence proof for the fact that *every* nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers. We provided a simple proof of this result plus an effective algorithm for finding such representations in [4]. However, in both our proof and McDaniel's proof one of the powerful numbers in such a representation is always a perfect square, except possibly when $n \equiv 2 \pmod 4$. Recently, Vanden Enyden [6] proved that also in the $n \equiv 2 \pmod 4$ case, one of the powerful numbers is always a square. We established in [4] that every *even* integer is representable in infinitely many ways as a *proper nonsquare difference of powerfuls*; i.e., as a proper difference of two powerful numbers neither of which is a perfect square. At this time, the only odd integer known to have such a representation is the integer 1 (see [7]). It is the purpose of this paper to complete the task; viz., to prove that every odd integer greater than 1 (hence every integer) is a proper nonsquare difference of powerfuls, and to provide an algorithm for finding such representations. Therefore, this paper establishes the fact that every nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers where either one of the powerful numbers is a perfect square and the other is not, or neither one of them is a perfect square.

For other work done on powerful numbers we refer the reader to our list of references.

2. NONSQUARE POWERFUL NUMBERS

To prove our main result, we will need the following lemma, which we state without proof since it is immediate from the binomial theorem.

Lemma: If B is an integer which is not a perfect square and $(T + U\sqrt{B})^i = T_i + U_i\sqrt{B}$, then

$$T_i = \sum_{k=0} \binom{i}{2k} T^{i-2k} U^{2k} B^k \quad \text{and} \quad U_i = \sum_{k=1} \binom{i}{2k-1} T^{i+2k+1} U^{2k-1} B^{k-1},$$

[1] *This author's research was supported by N.S.E.R.C. Canada, grant #A8484.*

[2] *This author was a senior undergraduate mathematics student at the University of Calgary during the writing of this paper.*

where ( ) denotes the binomial coefficient.

We are now in a position to prove the main result.

Theorem: Every nonzero integer is representable in infinitely many ways as a proper difference of two powerful numbers neither of which is a perfect square.

Proof: For the case where n is even see [4], and for the case where $n = 1$ see [7]. This leaves the case where $n > 1$ is odd. We break the proof down into two parts. We note that it suffices to prove the result for *either n or $-n$*.

Case (i): $n \not\equiv 0 \pmod 5$

Let $D = rs$, where

$$r = (n^2 - 2n + 5)/4 \quad \text{and} \quad s = (n^2 + 2n + 5)/4.$$

Let $T = (n^2 + 3)/4$, then $T^2 - D = -1$. If $(T + \sqrt{D})^i = (T_i + U_i\sqrt{D})$, then

$$T_i^2 - U_i^2 D = \pm 1.$$

Therefore,

$$\pm n = n(T_i^2 - DU_i^2) = sF_i^2 - rE_i^2,$$

where

$$E_i = T_i + sU_i \quad \text{and} \quad F_i = T_i + rU_i.$$

Now we show that, for an appropriate choice of i, we can achieve $E_i \equiv 0 \pmod r$ and $F_i \equiv 0 \pmod s$. To see this, we invoke the Lemma to get

$$E_i \equiv T^i + siT^{i-1} \pmod r.$$

Since $n \not\equiv 0 \pmod 5$, r and s are relatively prime, so we may choose

$$i \equiv -T(s)^{-1} \pmod r$$

which guarantees that $E_i \equiv 0 \pmod r$. Similarly, by choosing

$$i \equiv -T(r)^{-1} \pmod s$$

we guarantee $F_i \equiv 0 \pmod s$.

In order to complete Case (i), it remains to show that E_i and F_i are relatively prime. Suppose that there is a prime p such that:

$$E_i = T_i + sU_i = pt \tag{1}$$

for some integer t, and

$$F_i = T_i + rU_i = pu \tag{2}$$

for some integer u. Multiplying (1) by T_i and (2) by sU_i, then subtracting, we get

$$\pm 1 = T_i^2 - rsU_i^2 = p(tT_i - suU_i)$$

a contradiction.

Case (ii): $n \equiv 0 \pmod 5$

Let $D = n^2 + 1$, $T = n$, $U = -1$, and $(T + U\sqrt{D})^i = T_i + U_i\sqrt{D}$. Let $A_i = T_i + U_iD$ and $B_i = T_i + U_i$. Our plan of attack for this case is to show that for an appropriate choice of i we get $A_i^2 - B_i^2D = n^2$ with $(A_i \pm n)/2$ being powerful. First we observe that if $B_i \equiv 0 \pmod{2D}$ and g.c.d. $(A_i, n) = 1$, then $(A_i \pm n)/2$ are powerful. We prove g.c.d. $(A_i, n) = 1$ by contradiction. If there is a prime p such that $A_i = T_i + U_iD \equiv 0 \pmod p$ and $n \equiv 0 \pmod p$, then $T_i + U_i \equiv 0 \pmod p$. Therefore,

$$\pm 1 = T_i^2 - U_i^2D \equiv T_i^2 - U_i^2 \equiv 0 \pmod p,$$

a contradiction. Now, by choosing $i \equiv n \pmod{2D}$, we get by the Lemma that:

$$B_i = T_i + U_i \equiv T^i - iT^{i-1} \equiv 0 \pmod{2D}.$$

Hence, we have shown that $(A_i \pm n)/2$ are powerful. It remains to show that neither of these is a perfect square. To do this, we use the following fact. Since $n \equiv 0 \pmod 5$, D must contain, in its prime decomposition, a prime $p > 2$ to an odd exponent; i.e., $D \neq 2d^2$ for any integer d.

We observe that $A_i^2 - n^2 = B_i^2D = 2^5ef^2$, where e is odd. Therefore, whichever of $(A_i \pm n)/2$ is even cannot be a perfect square. It remains to show that $(A_i + n) \not\equiv 0 \pmod{4p}$ and $(A_i - n) \not\equiv 0 \pmod{4p}$; i.e., whichever of $(A_i \pm n)/2$ is odd cannot be a perfect square, since it contains the odd power of p.

Suppose $A_i + n \equiv 0 \pmod{4p}$. Therefore, $T_i + U_iD + n \equiv 0 \pmod{4p}$, which implies

$$T_i \equiv n^i \equiv -n \pmod p.$$

Hence, $n^{i-1} \equiv -1 \equiv n^2 \pmod p$, which implies

$$i \equiv 3 \equiv n \pmod 4.$$

Now, by the Lemma, $T_i \equiv 1 \pmod 4$ and $U_i \equiv 3 \pmod 4$. Thus,

$$0 \equiv T_i + U_iD + n \equiv 1 + 6 + 3 \pmod 4,$$

a contradiction.

Finally, assume $A_i - n \equiv 0 \pmod{4p}$. Therefore, $T_i + U_iD - n \equiv 0 \pmod{4p}$, which implies $T_i \equiv n^i \equiv n \pmod p$, and so $i \equiv 1 \equiv n \pmod 4$. Hence,

$$0 \equiv T_i + U_iD - n \equiv 1 + 6 - 1 \pmod 4,$$

a contradiction which secures the Theorem.

We note that the proof of the Theorem yields an effective algorithm, via the choice of i, for infinitely many representations of a given odd integer as a proper nonsquare difference of powerful numbers. The following examples illustrate the process.

Example 1: Let $i \equiv 1 \pmod{10}$ and $(3 + \sqrt{10})^i = T_i + U_i\sqrt{10}$. Thus,

$$3 = 2(T_i + 5U_i)^2 - 5(T_i + 2U_i)^2$$

with $T_i + 2U_i \equiv 0 \pmod 5$ and $T_i + 5U_i$ even. In particular, if $i = 1$, then $3 = 2^7 - 5^3$.

Example 2: Let $i \equiv 5 \pmod{52}$ and $(5 - \sqrt{26})^i = T_i + U_i\sqrt{26}$. Then

$$(T_i + 26U_i)^2 - 26(T_i + U_i)^2 = 25$$

with $(T_i + 26U_i \pm 5)/2$ nonsquare powerful numbers. In particular, if $i = 5$, then $5 = 7^2 \cdot 13^3 - 2^7 \cdot 29^2$.

REFERENCES

1. S. W. Golomb. "Powerful Numbers." *Amer. Math. Monthly* **77** (1970):848-852.
2. A. Makowski. "On a Problem of Golomb on Powerful Numbers." *Amer. Math. Monthly* **79** (1972):761.
3. W. L. McDaniel. "Representations of Every Integer as the Difference of Powerful Numbers." *The Fibonacci Quarterly* **20**, no. 1 (1982):85-87.
4. R. A. Mollin & P. G. Walsh. "On Powerful Numbers." (To appear: *Internat. J. Math. & Math. Sci.*)
5. W. A. Sentance. "Occurrences of Consecutive Odd Powerful Numbers." *Amer. Math. Monthly* **88** (1981):272-274.
6. C. Vanden Eynden. "Differences between Squares and Powerful Numbers." *The Fibonacci Quarterly* **24**, no. 4 (1986):347-348.
7. D. T. Walker. "Consecutive Integer Pairs of Powerful Numbers and Related Diophantine Equations." *The Fibonacci Quarterly* **14**, no. 2 (1976):111-116.

◆◇◆◇◆