THE LENGTH OF A TWO-NUMBER GAME

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1. INTRODUCTION

Let D be an operator defined on a pair of integers

$$A = (a_1, a_2),\ a_1 \geqslant a_2 > 0,$$

by

$$D(a_1, a_2) = \begin{cases} (a_2, a_1 - a_2), & 2a_2 \geqslant a_1, \\ (a_1 - a_2, a_2), & a_1 \geqslant 2a_2. \end{cases} \tag{1.1}$$

Given any initial pair A_0, we obtain a sequence $\{A_n\}$ with $A_n = DA_{n-1}$, $n > 0$. This sequence is called the "two-number game."

Definition 1.1: The length of the sequence $\{A_n\}$, denoted $L(A)$, is n such that $A_n = (a', 0)$ for some integer $a' > 0$.

Definition 1.2: The complement of A is $CA = (a_1, a_1 - a_2)$.

It follows that $C^2A = A$ and

$$DCA = DA. \tag{1.2}$$

The effect of D on (a_1, a_2) is to reduce a_1 by a_2 and then arrange $a_1 - a_2$ and a_2 in order of decreasing magnitude to form $D(a_1, a_2)$.

The number pair (a_1, a_2) may be replaced by a rectangle $(a_1 . a_2)$ of sides a_1 and a_2. In such a case, $D(a_1 . a_2)$, $C(a_1 . a_2)$, and $L(a_1 . a_2)$ may be defined as above, but by replacing the comma with a dot. $D(a_1 . a_2)$ and $C(a_1 . a_2)$ are then rectangles. The length $L(a_1 . a_2)$ is equal to the number of squares obtained by removing the largest square $(a_1 . a_2)$ from an end of $(a_1 . a_2)$, then the largest square from an end of the remaining rectangle, and so on, until no squares remain. Therefore,

$$L(a_1, a_2) = L(a_1 . a_2). \tag{1.3}$$

For example,

$$\begin{aligned}(5 . 3) &= (3 . 3) + (3 . 2) = (3 . 3) + (2 . 2) + (2 . 1) \\ &= (3 . 3) + (2 . 2) + (1 . 1) + (1 . 1)\end{aligned}$$

from which $L(5 . 3) = 4$. See Figure 1 on page 175.

Replace (a_1, a_2) by the vector $A = \binom{a_1}{a_2}$, and write D in matrix form:

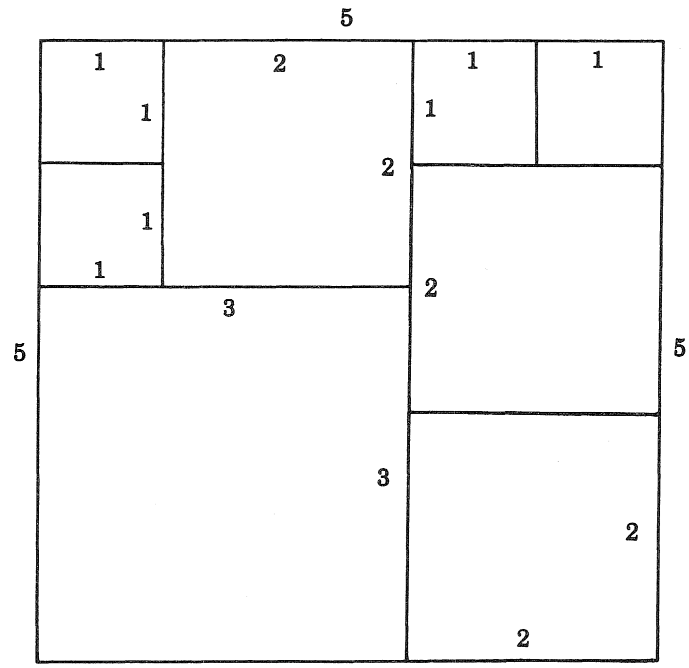

FIGURE 1. $L(5 \,.\, 3) = L(5 \,.\, 2) = 4$, $C(5 \,.\, 3) = (5 \,.\, 2)$

$$DA = \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} A, & 2a_2 \geqslant a_1, \\ \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A, & a_1 \geqslant 2a_2. \end{cases} \tag{1.4}$$

Then $DkA = kDA$ for $k > 0$, and

$$L(kA) = L(A). \tag{1.5}$$

It follows from the definition that

$$L\begin{pmatrix} a_1 + na_2 \\ a_2 \end{pmatrix} = n + L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \; n > 0. \tag{1.6}$$

Choose c such that $a_2 \mid (a_1 - c)$ and $a_1 > a_2 > c > 0$. Then,

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \dfrac{(a_1 - c)}{a_2} a_2 + c \\ a_2 \end{pmatrix},$$

and from (1.6),

$$L\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{a_1 - c}{a_2} + L\begin{pmatrix} a_2 \\ c \end{pmatrix}. \tag{1.7}$$

Now, $(a_1 - c)/a_2$ is the greatest integer in a_1/a_2, since a_2 divides $a_1 - c$ and $a_2 > c > 0$, so

$$\frac{a_1 - c}{a_2} = \left[\frac{a_1}{a_2}\right],$$

where $[x]$ represents the greatest integer function of x. Since c represents the quantity $a_1 \pmod{a_2}$, Equation (1.7) may be written

$$L\binom{a_1}{a_2} = \left[\frac{a_1}{a_2}\right] + L\binom{a_2}{a_1 \pmod{a_2}}. \tag{1.8}$$

This relation may be iterated as in the following example:

$$L\binom{23}{5} = \left[\frac{23}{5}\right] + \left[\frac{5}{3}\right] + \left[\frac{3}{2}\right] + \left[\frac{2}{1}\right] = 8.$$

Table 1 exhibits $L\binom{a_1}{a_2}$ for a_1, a_2 equal to 1, 2, ..., 15.

TABLE 1. $L\binom{a_1}{a_2}$

a_2 \ a_1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2		1	3	2	4	3	5	4	6	5	7	6	8	7	9
3			1	4	4	2	5	5	3	6	6	4	7	7	5
4				1	5	3	5	2	6	4	6	3	7	5	7
5					1	6	5	5	6	2	7	6	6	7	3
6						1	7	4	3	4	7	2	8	5	4
7							1	8	6	6	6	6	8	2	9
8								1	9	5	6	3	6	5	9
9									1	10	7	4	7	7	4
10										1	11	6	7	5	3
11											1	12	8	7	7
12												1	13	7	5
13													1	14	9
14														1	15
15															1

Let

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \text{ and } P = CQ = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

From (1.4), we have two forms of D^{-1}: $D_0^{-1} = Q$ and $D_1^{-1} = P$. D^{-2} has 2^2 forms, namely Q^2, QP, PQ, and P^2. D^{-n} has 2^n forms called D_j^{-n} which are the terms in the expansion of $(Q + P)^n$, where P and Q do not commute. The 2^n numbers $j = 0, 1, 2, \ldots, 2^n - 1$ may be expressed uniquely in binary form using n digits so that each D_j^{-n} may be paired with a distinct binary number.

Definition 1.3: We choose to define D_j^{-n} as the product derived from the binary number j of n digits in which 0 is replaced by Q and 1 by P.

For example, if $j = 3$, $n = 4$, the binary form of j is 0 0 1 1, so that $D_3^{-4} = Q^2P^2$.

It follows that $D^{-1}D^{-n} = D^{-n-1}$ and

$$D_i^{-m}D_j^{-n} = D_k^{-m-n}, \text{ where } k = 2^n i + j. \tag{1.9}$$

Note that D_i^{-m} and D_j^{-n} do not commute.

2. SEQUENCES OF VECTORS

Definition 2.1: If $a_1 \geqslant a_2$, A is said to be *proper*, and if a_1 and a_2 are relatively prime, then A is said to be *prime*.

We will assume henceforth that A is a proper prime vector. It follows that PA and QA are proper prime vectors, and hence $D^{-n}A$ in any of its forms is proper and prime.

Definition 2.2: Let $A(i, j)$ represent the vector A of length $i = L(A)$ as follows:

$$A(1, 0) = DA(2, 0) = \binom{1}{1},$$

$$A(2, 0) = D^0A(2, 0) = \binom{2}{1},$$

$$A(3, 0) = D_0^{-1}A(2, 0) = \binom{3}{2},$$

$A(3, 1) = D_1^{-1}A(2, 0) = \binom{3}{1}$ and if $i > 2$, $j = 0, 1, 2, \ldots, 2^{i-2} - 1$,

$$A(i, j) = D_j^{-i+2}A(2, 0).$$

Consider the sequence $\{X_n = A(n + 2, j),\ n = 1, 2, \ldots\}$, where

$$X_n = D_j^{-n}\binom{2}{1} \text{ and } L(X_n) = n + 2.$$

If $j = 0$, then

$$X_n = Q^n\binom{2}{1} \text{ and } X_{n+2} - X_{n+1} - X_n = 0 \text{ from the identity } Q^2 - Q - I = 0.$$

This identity may also be applied to cases where $j = 2^{n-1}$, 1, and $2^{n-1} + 1$ to yield the same recurrence relation. If $j = 2^n - 1$,

$$X_n = P^n\binom{2}{1} \text{ and } X_{n+2} - 2X_{n+1} + X_n = 0 \text{ from the identity } P^2 - 2P + I = 0.$$

This relation also holds for $j = 2^{n-1} - 1$, where $X_n = QP^{n-1}\binom{2}{1}$.

Note that X is represented as a product of elements selected from the set (P, Q) and a vector $\binom{2}{1}$. Then CX_n is X_n in which its first matrix (P or Q) is replaced by its complement (Q or P). X_n and CX_n have the same recurrence relations. See Table 2.

TABLE 2. Sequences $\{X_n = A(n+2, j)\}$

j	X_n	Recurrence
0	$Q^n\binom{2}{1} = \binom{F_{n+3}}{F_n}$	$X_{n+2} - X_{n+1} - X_n = 0$
2^{n-1}	$PQ^{n-1}\binom{2}{1} = \binom{F_{n+3}}{F_{n+1}}$	$X_{n+2} - X_{n+1} - X_n = 0$
1	$Q^{n-1}P\binom{2}{1} = \binom{L_{n+1}}{L_n}$	$X_{n+2} - X_{n+1} - X_n = 0$
$2^{n-1}+1$	$PQ^{n-2}P\binom{2}{1} = \binom{L_{n+1}}{L_{n-1}}$ if $n > 1$	$X_{n+2} - X_{n+1} - X_n = 0$
$2^n - 1$	$P^n\binom{2}{1} = \binom{n+2}{1}$	$X_{n+2} - 2X_{n+1} + X_n = 0$
$2^{n-1} - 1$	$QP^{n-1}\binom{2}{1} = \binom{n+2}{n+1}$	$X_{n+2} - 2X_{n+1} + X_n = 0$

Let $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, $X_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix}$, and $KX_i = X_{i+1}$, $i = 0, 1, 2, \ldots$, so that

$$K^nX_0 = X_n. \tag{2.1}$$

The characteristic equation for k is $|yI - K| = 0$ or

$$y^2 - (k_{11} + k_{22})y + |K| = 0.$$

By the Cayley-Hamilton theorem,

$$K^2 - (k_{11} + k_{22})K + |K|I = 0.$$

Multiply both sides of this equation on the right by $K^{n-2}X_0$, then

$$K^nX_0 - (k_{11} + k_{22})K^{n-1}X_0 + |K|K^{n-2}X_0 = 0.$$

From Equation (2,1),

$$X_n = (K_{11} + K_{22})X_{n-1} - |K|X_{n-2}, \tag{2.2}$$

a recurrence relation for X_n. We will assume here that

$$X_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

The sequences $\{x_{n1}\}$ and $\{x_{n2}\}$ have been described by Horadam [1] as

$$\{w_n\} = \{w_n(a, b; p, q)\} : w_0 = a,\ w_1 = b,\ w_n = pw_{n-1} - qw_{n-2},\ n \geqslant 2.$$

In either sequence, $p = \mathrm{tr}(K)$, the trace of K, and $q = |K|$.

We may substitute D_j^{-r} for K and $A(2,0)$ for X_0 in (2.1) to yield a sequence with the property $L(X_n) = rn + 2$. Let $D_j^{-r} = S_1S_2 \ldots S_r$, where $S_i \in (P, Q)$.

Note that any 2×2 matrices A and B have the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, so

$$\mathrm{tr}(S_1S_2 \ldots S_r) = \mathrm{tr}(S_2S_3 \ldots S_rS_1).$$

Therefore, p is the same for K equal to any cyclic product of the S_i. Since

$$|P| = 1 \quad \text{and} \quad |Q| = -1,$$

$q = |K| = (-1)^s$, where s represents the number of S_i equal to Q. Consider the example:

$$D_{10}^{-5} = QPQPQ = \begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}.$$

There are five different cyclic products of the S_i: $j = 5, 9, 10, 18, 20$. These form the sequences

$$\{D_j^{-5n}A(2, 0) = X_n : n = 0, 1, 2, \ldots\}$$

having the recurrence relation

$$X_n = 9X_{n-1} + X_{n-2}$$

and satisfying $L(X_n) = 5n + 2$. These sequences are exhibited in Table 3.

TABLE 3. Related Sequences

j	D_j^{-5}	$\{X_n : n = 0, 1, 2, \ldots\}$
5	$\begin{pmatrix} 7 & 5 \\ 3 & 2 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 19 \\ 8 \end{pmatrix}, \begin{pmatrix} 173 \\ 73 \end{pmatrix}, \ldots\right\}$
9	$\begin{pmatrix} 8 & 3 \\ 3 & 1 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 19 \\ 7 \end{pmatrix}, \begin{pmatrix} 173 \\ 64 \end{pmatrix}, \ldots\right\}$
10	$\begin{pmatrix} 7 & 3 \\ 5 & 2 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 12 \end{pmatrix}, \begin{pmatrix} 155 \\ 109 \end{pmatrix}, \ldots\right\}$
18	$\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 15 \\ 11 \end{pmatrix}, \begin{pmatrix} 137 \\ 100 \end{pmatrix}, \ldots\right\}$
20	$\begin{pmatrix} 5 & 7 \\ 3 & 4 \end{pmatrix}$	$\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 17 \\ 10 \end{pmatrix}, \begin{pmatrix} 155 \\ 91 \end{pmatrix}, \ldots\right\}$

REFERENCE

1. A. F. Horadam. "Basic Properties of a Certain Sequence of Numbers." *The Fibonacci Quarterly* 3, no. 3 (1965):161-76.

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