# RATIOS OF GENERALIZED FIBONACCI SEQUENCES

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## INTRODUCTION

In addition to the well-known Fibonacci sequence $F(n)$, recursively defined by

$$F(1) = 1,\ F(2) = 1,\ F(n+1) = F(n) + F(n-1), \text{ for } n > 2,$$

is the Lucas sequence $L(n)$, similarly defined by

$$L(1) = 1,\ L(2) = 3,\ L(n+1) = L(n) + L(n-1).$$

Although the difference $L(n) - F(n)$ increases without bound, the ratio $L(n)/F(n)$ tends to a limiting value of $\sqrt{5}$. This result follows from the two representations:

$$F(n) = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n - \frac{1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n;\ L(n) = \left[\frac{1+\sqrt{5}}{2}\right]^n + \left[\frac{1-\sqrt{5}}{2}\right]^n.$$

For a given integer $m \geqslant 3$, we now consider the sequence $G_m(n)$ defined by

$$G_m(1) = 1,\ G_m(2) = m,\ G_m(n+1) = G_m(n) + G_m(n-1).$$

From this we have $G_m(n) = L(n) + (m-3)F(n-1)$ and, consequently, the ratio $G_m(n)/F(n)$ has a limiting value of $\sqrt{5} + (m-3)(\sqrt{5}-1)/2$. This relationship also holds for any integral $m$ since the inequality $m \geqslant 3$ was not crucial to the validity of the statement. Indeed, the result is valid for all real $m$.

For Fibonacci-type sequences that begin with a nonzero first term other than one, say, for example, the sequence $H_{a,b}(n)$ defined by

$$H_{a,b}(1) = a,\ H_{a,b}(2) = b,\ H_{a,b}(n+1) = H_{a,b}(n) + H_{a,b}(n-1),$$

each term of which is merely a constant multiple of a $G_m$ sequence, namely,

$$H_{a,b}(n) = aG_{b/a}(n).$$

This means that the ratio $H_{a,b}(n)/F(n)$ has a limiting value of

$$a[\sqrt{5} + (b/a - 3)(\sqrt{5} - 1)/2].$$

Finally, for real numbers $a$, $b$, $c$, and $d$ with $ac \neq 0$, the ratio of $H_{a,b}(n)$ to $H_{c,d}(n)$ has a limiting value shown by

$$\frac{H_{a,b}(n)}{H_{c,d}(n)} = \frac{H_{a,b}(n)/F(n)}{H_{c,d}(n)/F(n)} \to \frac{2a\sqrt{5} + (b-3a)(\sqrt{5}-1)}{2c\sqrt{5} + (d-3c)(\sqrt{5}-1)}. \tag{1}$$

## GENERALIZED SEQUENCES

Let us consider the more general case where a Fibonacci-type sequence $F_k$ is recursively defined by the sum of the previous $k$ terms. The first $k$ terms are arbitrarily defined by $F_k(0) = a_1$, $F_k(1) = a_2$, ..., $F_k(k-1) = a_k$, and then

$$F_k(i) = \sum_{j=i-k}^{i-1} F_k(j), \text{ for } i \geqslant k.$$

From the theory of recursion we know that $F_k$ is generated by a finite $k$-sum of powers by

$$F_k(n) = f_1 r_1^n + f_2 r_2^n + \cdots + f_k r_k^n,$$

where $f_i$ are constants (real or complex), and the $r_i$ are the zeros of the polynomial

$$p(x) = x^k - x^{k-1} - \cdots - x - 1. \tag{2}$$

It is shown in [3] that the roots of $p$ are all distinct, and all lie within the unit circle in the complex plane except one root which is real and lies between 1 and 2. For simplicity this real root will be labeled $r_k$, and the others will be denoted by $r_1, r_2, \ldots, r_{k-1}$. This means $|r_i| < 1$ for $i < k$, and $1 < r_k < 2$.

The graphs of $p$ for various $k$ help to illustrate the location of the roots, as well as the additional fact that $r_k \to 2$ as $k$ increases [2]. It is important to realize that these roots are determined as soon as $k$ is known, and that they have nothing to do with the initial values given to $F_k(0), F_k(1), \ldots, F_k(k-1)$.

The constants $f_i$ can be determined from the side conditions $a_i = F_k(i-1)$, and by applying Cramer's rule we get:

$$f_i = \frac{\begin{vmatrix} 1 & 1 & & 1 & a_1 & 1 & & 1 \\ r_1 & r_2 & & r_{i-1} & a_2 & r_{i+1} & & r_k \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ r_1^{k-1} & r_2^{k-1} & & r_{i-1}^{k-1} & a_k & r_{i+1}^{k-1} & & r_k^{k-1} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & & 1 \\ r_1 & r_2 & & r_k \\ \vdots & \vdots & \cdots & \vdots \\ r_1^{k-1} & r_2^{k-1} & & r_k^{k-1} \end{vmatrix}}. \tag{3}$$

Since the denominator of this expression is the $k \times k$ Vandermonde determinant, its value is given by

$$\prod_{\substack{i=2 \\ i>j}}^{k} (r_i - r_j). \tag{4}$$

Suppose we have two such Fibonacci sequences of the same type, say $F_k$ and $G_k$, where

$$F_k(i) = a_{i+1} \quad \text{and} \quad G_k(i) = b_{i+1} \text{ for } 0 \leqslant i \leqslant k-1.$$

Then there exist constants $f_1, f_2, \ldots, f_k, g_1, g_2, \ldots, g_k$ such that

$$F_k(n) = \sum_{i=1}^{k} f_i r_i^n \quad \text{and} \quad G_k(n) = \sum_{i=1}^{k} g_i r_i^n \tag{5}$$

and the $r_i$ are the roots to (2). The ratio $F_k(n)/G_k(n)$ must then approach $f_k/g_k$ as $n$ increases. The problem then becomes one to evaluate $f_k$ and $g_k$ which, in turn, reduces to solving $p(x) = 0$.

## TWO SPECIFIC CASES

When $k = 2$, we have $F_2(n) = f_1r_1^n + f_2r_2^n$ and $G_2(n) = g_1r_1^n + g_2r_2^n$, and the ratio $F_2(n)/G_2(n) \to f_2/g_2$ where, from (3), we get

$$f_2 = \frac{a_2 - a_1r_1}{r_2 - r_1} \quad \text{and} \quad g_2 = \frac{b_2 - b_1r_1}{r_2 - r_1}. \tag{6}$$

Since $r_1$, $r_2$ are the roots to $x^2 - x - 1 = 0$, with $r_2$ being the root of modulus between 1 and 2, then $r_1 = (1 - \sqrt{5})/2$. Thus, the ratio $f_2/g_2$ reduces to

$$\frac{2a_2 - a_1(1 - \sqrt{5})}{2b_2 - b_1(1 - \sqrt{5})}, \tag{7}$$

and this agrees with our earlier result from (1).

For $k = 3$, $F_3(n) = f_1r_1^n + f_2r_2^n + f_3r_3^n$ and $G_3(n) = g_1r_1^n + g_2r_2^n + g_3r_3^n$, and then

$$f_3 = \frac{a_1r_1r_2 - a_2(r_1 + r_2) + a_3}{(r_3 - r_2)(r_3 - r_1)} \quad \text{and} \quad g_3 = \frac{b_1r_1r_2 - b_2(r_1 + r_2) + b_3}{(r_3 - r_2)(r_3 - r_1)}$$

so then

$$f_3/g_3 = \frac{a_1r_1r_2 - a_2(r_1 + r_2) + a_3}{b_1r_1r_2 - b_2(r_1 + r_2) + b_3}. \tag{8}$$

The values for $r_1$, $r_2$ are determined by using Cardano's formula:

$$r_1 = \frac{1}{6}\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} + 3i\left\{\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right\}\right]$$

$$r_2 = \frac{1}{6}\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} - 3i\left\{\sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right\}\right].$$

This gives the approximate values $r_1 = -.4196433 + .6062906i$ and $r_2 = \overline{r}_1$. Consequently, the ratio $f_3/g_3$ is real (since $r_1r_2$ and $r_1 + r_2$ are real) with the approximate value

$$f_3/g_3 = \frac{.5436888a_1 + .8392866a_2 + a_3}{.5436888b_1 + .8392966b_2 + b_3}. \tag{9}$$

Evaluation of $f_k/g_k$ for $k > 3$ ultimately rests on effectively computing the complex roots to $p(x) = 0$.

## APPROXIMATING COMPLEX ROOTS

Among the many iterative numerical methods available for locating roots to polynomial equations, probably the best known is Newton's method. Typically, this method is employed to find real roots, but it can be generalized to the complex plane [5]. To this end, we begin with a complex seed $z_0$, and consider the sequence $\{z_n\}$ of iterates, $z_{n+1} = z_n - p(z_n)/p'(z_n)$. It appears, from data gathered, that every complex seed generates a sequence that eventually converges to a root of $p(z)$ with, of course, varying rates of convergence. But an interesting question, and one that was posed as far back as 1879 by Arthur Cayley [4], is to determine the regions of the plane whose members generate sequences that converge to identical roots of $p(z)$. The readers may wish to determine the corresponding regions for a specific polynomial. The author gathered data on $z^3 - z^2 - z - 1 = 0$ and approximated the partitions of

$\{(x, y) : |x| \leqslant 1, |y| \leqslant 1\}$.

The shaded regions in Figure 1 consist of those "seeds" that generate sequences that converge to the root $r_2$ with $r_2 = -.4196 - .6063i$. Obviously there is no reason to suspect that the points in the plane that generate sequences that converge to the same root form a connected set. Likewise, statements concerning symmetry of regions are not obvious to formulate. Instead, there is some considerable disconnectedness to the regions, especially for this one in the near vicinity of the $x$-axis, where one can find seeds that generate sequences that converge to each of the three roots to the polynomial.

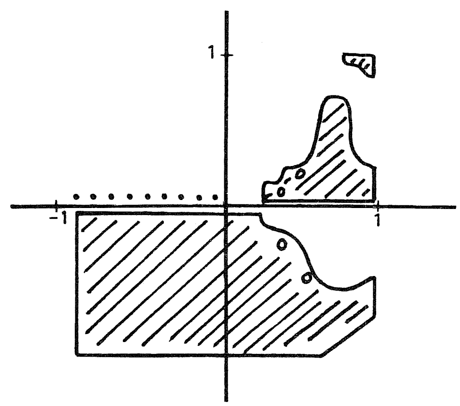


FIGURE 1. A region whose members generate the same polynomial root

It is of interest to point out that the associated notion of Julia sets (a concept developed by Julia and Fatou at the turn of the century in regard to the iteration of rational functions in the plane) is discussed in [4] and accompanied by some excellent color computer graphics.

## CONSECUTIVE FIBONACCI NUMBERS

Suppose we take a more careful look at the sequence of ratios of consecutive Fibonacci numbers. For the standard Fibonacci sequence $F(n)$, the sequence of ratios $F(n)/F(n-1)$ alternates monotonically. Thus, setting

$$r(n) = F(n)/F(n-1),$$

we have

$$r(2i) < r(2i+2),\ r(2i+1) > r(2i+3),\ r(2i) < r(2i+1), \text{ for all } i.$$

But what happens if $F(n)$ is replaced by the more general Fibonacci sequence $F_k(n)$, where

$$F_k(i) = a_i, \qquad \text{for } 1 \leqslant i \leqslant k,$$

$$F_k(i) = \sum_{j=i-k}^{i-1} F_k(j), \text{ for } i \geqslant k+1.$$

In this general setting the sequence of ratios $r_k(n) = F_k(n)/F_k(n-1)$ does not alternate monotonically, nor does it alternate in $k$-tuples. Patterns seem to be haphazard at best. But one can make a statement about the maximum number of ratios that form a consecutive monotone string. More specifically, this means (monotone increasing is sufficient)

$$\max\{j : \exists i,\ i > k \text{ and } r_k(i+1) < r_k(i+2) < \cdots < r_k(i+j)\} \leqslant k. \tag{10}$$

This inequality will be established if we show that whenever

$$\frac{F_k(i+1)}{F_k(i)} < \frac{F_k(i+2)}{F_k(i+1)} < \cdots < \frac{F_k(i+k)}{F_k(i+k-1)} \tag{11}$$

it follows that

$$\frac{F_k(i+k)}{F_k(i+k-1)} > \frac{F_k(i+k+1)}{F_k(i+k)}. \tag{12}$$

Setting $f_j = F_k(i+j)$ to simplify notation, it follows that

$$f_1 f_{k-1} < f_0 f_k,\ f_2 f_{k-1} < f_1 f_k,\ \ldots,\ f_{k-1} f_{k-1} < f_{k-2} f_k,$$

so summing gives

$$\sum_{i=1}^{k-1} f_i f_{k-1} < \sum_{i=0}^{k-2} f_i f_k \tag{13}$$

and then adding $f_{k-1} f_k$ to both summations yields

$$f_{k+1} f_{k-1} = \sum_{i=1}^{k} f_i f_{k-1} < \sum_{i=0}^{k-1} f_i f_k = f_k f_k, \tag{14}$$

which establishes the desired result.

So for each given choice of $k$, each string of ratios of consecutive $k$-generalized Fibonacci numbers $F_k(n)/F_k(n-1)$ will contain a maximum of $k$ consecutive monotone terms. Consider the following example with $k = 3$.

TABLE 1. Generalized Fibonacci Numbers and Their Ratios

| $n$ | $F_3(n)$ | $F_3(n)/F_3(n-1)$ |
|---|---|---|
| 1 | 1 | |
| 2 | 1 | |
| 3 | 2 | |
| 4 | 4 | 2.00 |
| 5 | 7 | 1.75 |
| 6 | 13 | 1.85714 |
| 7 | 24 | 1.84615 |
| 8 | 44 | 1.83333 |
| 9 | 81 | 1.84091 |

Here we have the three consecutive monotone terms,

$$\frac{F_3(6)}{F_3(5)} > \frac{F_3(7)}{F_3(6)} > \frac{F_3(8)}{F_3(7)} \tag{15}$$

and, of course, the next ratio reverses the monotonicity,

$$\frac{F_3(8)}{F_3(7)} < \frac{F_3(9)}{F_3(8)}. \tag{16}$$

Data seem to indicate that this result is best, in the sense that every Fibonacci sequence $F_k$ contains a string of exactly $k$ consecutive monotone ratios. What we can prove here is the existence of a sequence, for each $k$, which satisfies this conjecture. Thus, for $k \geqslant 2$, we define the sequence $F_k$ by $F_k(n) = 1$ for $n < k$ and $F_k(k) = k$. Then

$$F_k(k+1) = 2k - 1,\ F_k(k+2) = 4k - 3,$$

$$F_k(k+3) = 8k - 7,\ F_k(k+4) = 16k - 15,$$

and the pattern continues up to

$$F_k(2k-1) = 2^{k-1}k - (2^{k-1} - 1) \quad \text{and} \quad F_k(2k) = 2^k k - (2^k - 1).$$

The pattern breaks with the next term for

$$F_k(2k+1) = (2^{k+1} - 2)k - (2^{k+1} - 2 - k).$$

The ratios $F_k(k+i)/F_k(k+i-1)$ form an increasing sequence for $i = 1, 2, \ldots, k$ because the inequality

$$\frac{2^n k - (2^n - 1)}{2^{n-1}k - (2^{n-1} - 1)} < \frac{2^{n+1}k - (2^{n+1} - 1)}{2^n k - (2^n - 1)}$$

holds for all $n \geqslant 1$. Furthermore, the string of increasing ratios is then reversed with the next ratio because

$$\frac{F_k(2k)}{F_k(2k-1)} > \frac{F_k(2k+1)}{F_k(2k)}. \tag{18}$$

It is interesting to look at the similar question of finding a Fibonacci sequence with $k$ consecutive decreasing ratios. Unlike the previous example, such a solution cannot be found by defining the initial $k$ terms in the sequence from among the elements $1, 2, \ldots, k$. We need to choose from a larger set of positive integers. Thus, for $k \geqslant 2$, we define the sequence $F_k$ by

$$F_k(1) = 1,\ F_k(2) = 2,\ F_k(3) = 4,\ \ldots,\ F_k(k-1) = 2^{k-2},\ \text{and}\ F_k(k) = 1.$$

For values of $i$ with $1 \leqslant i \leqslant k$, the term $F_k(k+i)$ has the value

$$F_k(k+i) = 2^{k+(i-2)} - (i-1)2^{i-2} \tag{19}$$

and, consequently,

$$\frac{F_k(k+i)}{F_k(k+i-1)} > \frac{F_k(k+i+1)}{F_k(k+i)}, \text{ for } i = 1, 2, \ldots, k-1. \tag{20}$$

Many other questions remain for the interested reader to investigate. Can one predict when these monotone strings of ratios of length $k$ will occur, or how often they will occur? Are there strings of length $i$ for each $i$ less than $k$ for each given sequence? Are there as many increasing strings as decreasing strings?

## REFERENCES

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