SECTIONS, GOLDEN AND NOT SO GOLDEN

PHILIP G. ENGSTROM
University of Regina, Regina, Saskatchewan, Canada S4S 0A2
(Submitted May 1985)

INTRODUCTION

The idea of the golden section is familiar to a wide audience. While many of the properties that arise from consideration of the golden section seem to be unique to it, they may belong to a much wider class of "sections." This paper presents the golden section and certain related ideas as special cases of such a wider class.

To provide the context for what follows and to introduce some notation, we include here a quick reference to the golden ratio, ϕ. Let a line segment $\overline{AB}$ be given and, for convenience, let its length $AB = 1$. If we determine a point C between A and B and such that $AB/AC = AC/CB$, we say the point C divides AB in golden section (see Fig. 1).

Figure 1

It is a simple matter to find the ratio $\phi = AB/AC$ that belongs to the golden section. If we set $x = AC$, then $CB = 1 - x$ and we have the requirement

$$\phi = 1/x = x/(1 - x), \; 0 < x < 1.$$

From this equation, we infer that $1/\phi = \phi - 1$, or

$$\phi^2 - \phi - 1 = 0. \tag{1}$$

From the quadratic equation, and since $x > 0$, we have

$$\phi = (\sqrt{5} + 1)/2 \doteq 1.61803.$$

The number ϕ has the interesting property that if we subtract the value 1 from it we obtain its reciprocal.

As readers of this journal know well, the golden ratio bears a relation to the Pentagon of Pythagoras, so much admired by the Greeks, to the golden rectangle where it gives the ratio of adjacent sides, to the logarithmic spiral, and to the Fibonacci numbers. Specifically, if F_k denotes the k^{th} Fibonacci number, then as $n \to \infty$, $F_{n+1}/F_n \to \phi$.

THE MODIFIED GOLDEN SECTION AND GOLDEN RATIO

In what follows we will develop some ideas similar to those alluded to above and growing out of a generalization of the definition of the golden section. So consider again a line segment $\overline{AB}$ of length $AB = 1$ and let C be a point between A and B and such that $AB/AC = \alpha^2 AC/CB$ with $\alpha > 0$. We can write this relationship as

$$\frac{AB}{\alpha AC} = \frac{\alpha AC}{CB}, \quad \alpha > 0.$$

Set $\psi_\alpha = AB/\alpha AC$ and let $AC = x$. Then $CB = 1 - x$, $\psi_\alpha = 1/\alpha x = \alpha x/(1 - x)$, and

$$\psi_\alpha^2 - \frac{1}{\alpha}\psi_\alpha - 1 = 0, \tag{2}$$

which is the analogue of equation (1). For convenience, we let β represent the reciprocal of ψ_α:

$$\beta = 1/\psi_\alpha = \alpha x = \frac{\sqrt{1 + 4\alpha^2} - 1}{2\alpha}.$$

Suppose now that we let $\alpha > 0$ be chosen and construct a rectangle $ABDF$ (see Fig. 2) whose sides are in the ratio $\psi_\alpha = 1/\beta = AB/BD$. A few simple calculations show that if from such a rectangle we remove the rectangle $ACEF$ whose sides AC and CE are in the ratio $1/\alpha$, i.e., $AC = x = \beta/\alpha$ and $CE = \beta$, then the remaining rectangle $BDEC$ has sides also in the ratio

$$\alpha x/(1 - x) = \beta/(1 - x) = \psi_\alpha.$$

Thus, as in the case of the golden rectangle, the two rectangles $ABDF$ and $BDEC$ are similar and, by continuation of the process described here, we can construct an infinite nested sequence $R_1, R_2, R_3, \ldots, R_n, \ldots$ of rectangles, all of which are mutually similar.

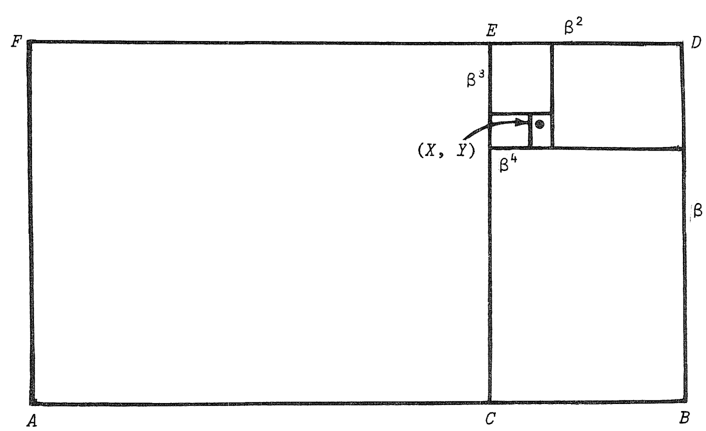

Figure 2

By varying α, of course, we vary the value of β and so also the shape of the rectangles. Since

$$\lim_{\alpha \to 0} \beta = \lim_{\alpha \to 0} \frac{\sqrt{1 + 4\alpha^2} - 1}{2\alpha} = 0,$$

It is clear that to small α there correspond small values of β. Thus, as $\alpha \to 0$ the rectangles tend toward "degenerate" rectangles, i.e., toward line segments.

From

$$\lim_{\alpha \to \infty} \beta = 1,$$

we infer that as α increases without bound the rectangles approach squares. We note also that for $0 < \alpha < \infty$, $0 < \beta < 1$.

Suppose that, for some value of α, we let $\{R_n\}$ be the associated sequence of rectangles obtained by the construction described above. Recall $R_{n-1} \supset R$. Take the vertex A (see Fig. 2) to be the origin of a rectangular coordinate system with the side $\overline{AB}$ lying on the positive x axis. Cantor's nested set theorem then assures us that there is a point (X, Y) which lies in each rectangle R_n. Now each rectangle R_n has sides of length β^{n-1} and β^n, with β^{n-1} being the longer side. It is clear from Figure 2 that

$$\begin{aligned} X &= 1 - \beta^2 + \beta^4 - \beta^6 + \cdots \\ &= 1/(1 - \beta^2) \qquad [= \alpha/(2\alpha - \beta)] \end{aligned}$$

and

$$\begin{aligned} Y &= \beta - \beta^3 + \beta^5 - \beta^7 + \cdots \\ &= \beta X \qquad [= \alpha\beta/(2\alpha - \beta)]. \end{aligned}$$

If we eliminate β from these equations, we find that

$$Y^2 = X - X^2 \quad \text{or} \quad \left(X - \frac{1}{2}\right)^2 + Y^2 = \left(\frac{1}{2}\right)^2.$$

Thus, the points (X, Y) lie on a circle of radius 1/2 and having its center at (1/2, 0). As $\alpha \to 0$, $(X, Y) \to (1, 0)$ along the circle, and as $\alpha \to \infty$, $(X, Y) \to (1/2, 1/2)$. Specifically, the points (X, Y) lie on the quarter-circle shown in Figure 3.

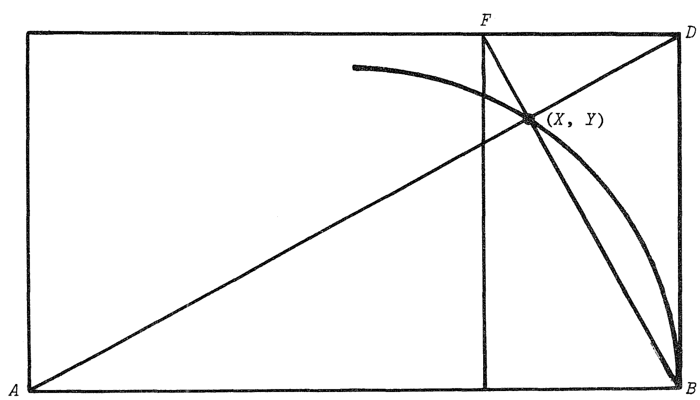

Figure 3

The point (X, Y) can be found by a very simple geometrical construction. If R_n and R_{n+1} are any two consecutive rectangles, then (X, Y) lies at the intersection of corresponding (and orthogonal) diagonals of these rectangles. Figure 3 above illustrates the case when R_1 and R_2 are the given rectangles. The diagonals here are $\overline{AD}$ and $\overline{BF}$.

We turn next to the logarithmic spiral associated with the rectangles R_n. Before doing so, however, we mention briefly the so-called rectangular spiral constructed from the longer sides of the rectangles R_n (see Fig. 4). This spiral "terminates," of course, at the point (X, Y) and has length (measured from the origin)

$$\begin{aligned} L &= 1 + \beta + \beta^2 + \cdots \\ &= 1/(1 - \beta) \\ &= \frac{2\alpha}{2\alpha - \sqrt{1 + 4\alpha^2} + 1} \\ &= \alpha(1 + \psi_\alpha). \end{aligned}$$

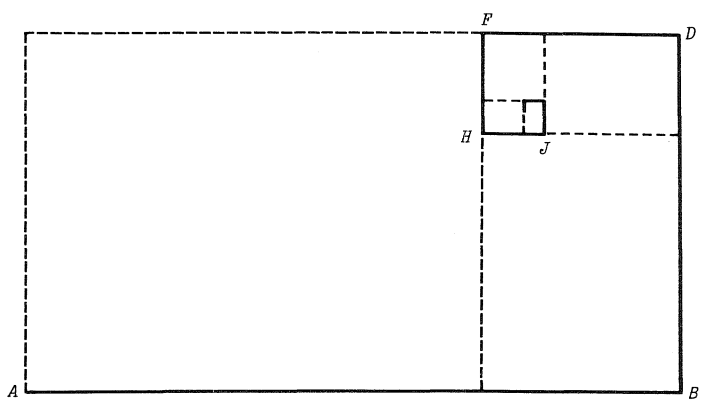

Figure 4

By a translation, we can place the origin of our coordinate system at the point (X, Y). Let P_n, $n = 1, 2, 3, \ldots$, be corresponding corners of the rectangles R_n and so also corners of the rectangular spiral. The points P_n can be shown, after some calculation, to have the following representations in terms of the new coordinate system:

$$\begin{aligned} P_1 &= (-X, -Y) \\ P_2 &= (1 - X, -Y) \\ P_3 &= (1 - X, \beta - Y) \\ P_4 &= (1 - \beta^2 - X, \beta - \beta^3 - Y) \end{aligned}$$

$$P_n = \begin{cases} \left(\dfrac{\beta^n}{1 + \beta^2}, \dfrac{-\beta^{n-1}}{1 + \beta^2}\right) & \text{if } n = 4k - 2 \\ \left(\dfrac{\beta^{n-1}}{1 + \beta^2}, \dfrac{\beta^n}{1 + \beta^2}\right) & \text{if } n = 4k - 1 \\ \left(\dfrac{-\beta^n}{1 + \beta^2}, \dfrac{\beta^{n+1}}{1 + \beta^2}\right) & \text{if } n = 4k \\ \left(\dfrac{-\beta^{n-1}}{1 + \beta^2}, \dfrac{-\beta^n}{1 + \beta^2}\right) & \text{if } n = 4k + 1. \end{cases}$$

If we express these points in terms of the polar coordinates r and θ, we obtain the cleaner expressions:

$$P_n = \left(\frac{\beta^{n-1}}{\sqrt{1+\beta^2}},\ \text{Arctan}\ \beta + ([n/2] - 1)\pi\right) \qquad \text{if } n \text{ is odd,}$$

or

$$P_n = \left(\frac{\beta^{n-1}}{\sqrt{1+\beta^2}},\ \text{Arctan}(-1/\beta) + ([n/2] - 1)\pi\right) \quad \text{if } n \text{ is even.}$$

Here $[\cdot]$ denotes the greatest integer function.

A few additional calculations convince us that each of the points P_n lies on a logarithmic spiral $r = ae^{b\theta}$. The constants a and b are easily determined from the requirement that the spiral pass through, let us say, P_2 and P_3. That it passes through P_2 implies that

$$r = \frac{\beta}{\sqrt{1+\beta^2}} \quad \text{and} \quad \theta = \text{Arctan}(-1/\beta) = -\text{Arctan}(1/\beta).$$

Thus,

$$\frac{\beta}{\sqrt{1+\beta^2}} = ae^{-b\ \text{Arctan}(1/\beta)}. \tag{3}$$

That the spiral passes through P_3 implies

$$r = \frac{\beta^2}{\sqrt{1+\beta^2}} \quad \text{and} \quad \theta = \text{Arctan}\ \beta.$$

Thus,

$$\frac{\beta^2}{\sqrt{1+\beta^2}} = ae^{b\ \text{Arctan}\ \beta}. \tag{4}$$

Combining equations (3) and (4) yields

$$\beta = e^{b(\text{Arctan}\ \beta + \text{Arctan}(1/\beta))}.$$

But the exponent here reduces to $b\pi/2$. So $\beta = e^{b\pi/2}$ and $b = 2\ln\beta/\pi < 0$. Now from (4) we can conclude that

$$a = \frac{\beta^2}{\sqrt{1+\beta^2}\ \exp(2\ \ln\ \beta\ \text{Arctan}\ \beta/\pi)}$$

and that

$$r = \frac{\beta^2}{\sqrt{1+\beta^2}\ \exp(2\ \ln\ \beta\ \text{Arctan}\ \beta/\pi)}\ \exp\left(\frac{2\ \ln\ \beta}{\pi}\theta\right).$$

Alternately,

$$r = \frac{1}{\sqrt{1+\beta^2}}\ \beta^{\frac{2(\theta + \pi - \text{Arctan}\ \beta)}{\pi}} \tag{5}$$

Figure 5 shows the spiral when $\alpha = 2$.

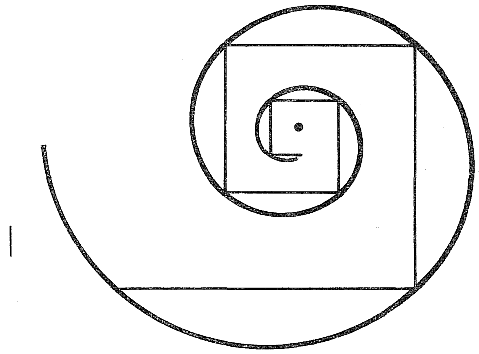

Figure 5

In the construction above, we have taken the points P_n to be at corresponding corners of the rectangles R_n. The decision to use these corners was arbitrary. If P_1 is chosen to be *any* point within or on R_1 and $P_2, P_3, \ldots$ to be the corresponding points of $R_2, R_3, \ldots$, then the spiral passing through all of the points P_n would again be logarithmic.

OTHER RELATED IDEAS

We consider next some relationships which are analogous to those between the golden section, golden rectangles, and the Fibonacci sequence. Let α be given and consider the sequence $\{u_n\}$, where

$$u_n = \frac{1}{\sqrt{1 + 4\alpha^2}}\left[\left(\frac{1 + \sqrt{1 + 4\alpha^2}}{2}\right)^n - \left(\frac{1 - \sqrt{1 + 4\alpha^2}}{2}\right)^n\right].$$

Readers familiar with the Fibonacci sequence will recognize that if $\alpha = 1$, then the last expression is the Binet formula and $\{u_n\}$ is nothing more than the Fibonacci sequence. To simplify calculations for the moment, set

$$z = \sqrt{1 + 4\alpha^2},\ a = (1 + z)/2, \text{ and } b = (1 - z)/2$$

so that $u_n = (a^n - b^n)/z$. Then it easily follows that

$$\begin{aligned} u_{n-1} + \alpha^2 u_{n-2} &= (1/z)[(a^{n-1} - b^{n-1}) + \alpha^2(a^{n-2} - b^{n-2})] \\ &= (1/z)[a^{n-2}(a + \alpha^2) - b^{n-2}(b + \alpha^2)]. \end{aligned}$$

But $a + \alpha^2 = a^2$ and $b + \alpha^2 = b^2$. Hence,

$$u_n = u_{n-1} + \alpha^2 u_{n-2}. \tag{6}$$

This serves as the law of generation for the sequence $\{u_n\}$. If $\alpha = 1$, this reduces to the familiar law of generation for the Fibonacci sequence.

Although we will not prove their validity, we list here a few of the relationships which are analogous to the relationships between terms of the Fibonacci sequence.

1. $u_n = (1 + 2\alpha^2)u_{n-2} - \alpha^4 u_{n-4}$
2. $u_1 + u_2 + u_3 + \cdots + u_n = (u_{n+2} - 1)/\alpha^2$
3. $u_1^2 + u_2^2 + \cdots + u_n^2 = (1 - \alpha^2)[u_1u_2 + u_2u_3 + \cdots + u_{n-1}u_n] + u_nu_{n+1}$
4. $u_n^2 - u_{n-1}u_{n+1} = (-\alpha^2)^{n-1}$
5. For any positive integer k, $u_n | u_{kn}$.

The first four of these relationships can be shown by an appeal to (6) and/or an induction proof. The fifth follows directly from the definition of u_n. From the second of these relationships, we can infer that when α is an integer, then $u_n \equiv 1 \bmod \alpha^2$. The table below gives values of u_n for some choices of α and of n.

$n =$	$\alpha =$ 1/3	1/2	2	3	4
1	1	1	1	1	1
2	1	1	1	1	1
3	1.11111	1.25	5	10	17
4	1.22222	1.50	9	19	33
5	1.345679	1.8125	29	109	205
6	1.4814814	2.1875	65	280	833
7	1.6310013	2.640625	181	1261	5713
8	1.7956104	3.18145	441	3781	19041
9	1.9768328	3.84765	1165	15130	110449
10	2.1763450	4.64453	2929	49159	415105
⋮			⋮	⋮	⋮
15			325525	28607050	884773585

Our next question is the obvious one:

How does $\lim_{n\to\infty} u_{n+1}/u_n$ relate to the ratio ψ_α?

From the definition of u_n, we can write

$$\frac{u_{n+1}}{u_n} = \frac{a^{n+1} - b^{n+1}}{a^n - b^n} = a\,\frac{1 - (b/a)^{n+1}}{1 - (b/a)^n}.$$

But $|b/a| = |(1 - z)/(1 + z)| < 1$, since $z > 0$ for all α. Thus,

$$\lim_{n\to\infty} u_{n+1}/u_n = a = (1 + z)/2 = \alpha\psi_\alpha.$$

This relationship is the analogue of

$$\lim_{n\to\infty} F_{n+1}/F_n = \phi,$$

where $\{F_n\}$ is the Fibonacci sequence and ϕ is the golden ratio.

Two further interesting properties which belong to the Fibonacci numbers also belong to the "modified Fibonacci numbers" u_n. If we form a matrix M of order $m \geqslant 3$ and whose entries, row by row, are m^2 successive terms u_k, u_{k+1}, u_{k+2}, ..., u_{k+m^2-1}, then det $M = 0$. So, for example, if $m = 3$ and $\alpha = 2$, and if we choose the nine successive terms of $\{u_n\}$ to be 1, 5, 9, 29, 65, 181, 441, 1165, and 2929, then

$$M = \begin{bmatrix} 1 & 5 & 9 \\ 29 & 65 & 181 \\ 441 & 1165 & 2929 \end{bmatrix}.$$

That the determinant of M is zero follows from the fact that, in any matrix M constructed as above from the successive terms u_n, the third column, U_3 (regarded here as a column vector), is equal to $U_2 + \alpha^2 U_1$, where U_1 and U_2 are the first and second column vectors belonging to M.

Another interesting property relates to magic squares of order 3. We illustrate this with an example. Thus, consider the magic square

8	1	6
3	5	7
4	9	2

Summing along rows, columns, or diagonals yields the same result, 15. Now let α be given and determine the terms u_1, u_2, u_3, ..., u_9 of the sequence $\{u_n\}$. In the magic square above, replace the number k with the term u_k to obtain the square

u_8	u_1	u_6
u_3	u_5	u_7
u_4	u_9	u_2

Then

$$u_8u_1u_6 + u_3u_5u_7 + u_4u_9u_2 = u_8u_3u_4 + u_1u_5u_9 + u_6u_7u_2.$$

For $\alpha = 3$, the above square with the associated products and sums is:

3781	1	280	1058680
10	109	1261	1374490
19	15130	1	287470
718390	1649170	35380	2720640

More generally, let the magic square

h	i	j
k	ℓ	m
p	q	r

be given, where

$$h + i + j = k + \ell + m = p + q + r = h + k + p = i + \ell + q = j + m + r.$$

Now, construct the corresponding square

u_h	u_i	u_j
u_k	u_ℓ	u_m
u_p	u_q	u_r

whose entries are the modified Fibonacci numbers.

Employing the notation of page 123 [$u_n = (a^n - b^n)/z$], it is a simple matter to show that

$$u_h u_i u_j + u_k u_\ell u_m + u_p u_q u_r = u_h u_k u_p + u_i u_\ell u_q + u_j u_m u_r.$$

The reader will quickly observe, for example, that the expansion of the expression $u_h u_i u_j$ contains the term $(1/z^3)a^{h+i+j}$, while the expansion of $u_h u_k u_p$ contains the term $(1/z^3)a^{h+k+p}$. But $h+i+j = h+k+p$. Similarly, the expansion of $u_k u_\ell u_m$ contains the term $(-1/z^3)a^{k+\ell}b^m$, while the expansion of $u_j u_m u_r$ contains the term $(-1/z^3)a^{j+r}b^m$. But $j + r = k + \ell$ so that the terms in question are equal.

While the property alluded to here holds for any 3×3 magic square, it does not hold generally for larger magic squares. The reader may verify this by considering the 4×4 magic square:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

CONCLUSION

We have provided here only a few of the most significant relationships and properties arising from consideration of the ratio ψ_α. Many others analogous to those arising from the golden ratio may be found. Indeed, what we have shown here places the golden section, the golden rectangles, the Fibonacci sequence, and the properties pertaining to them within a continuum in which they appear as a part of the special case $\alpha = 1$.

REFERENCES

1. Stanley Gudder. *A Mathematical Journey*. New York: McGraw-Hill, 1976.
2. H. E. Huntley. *The Divine Proportion*. New York: Dover, 1970.
3. N. N. Vorob'ev. *Fibonacci Numbers*. Popular Lectures in Mathematics, Vol. II (translated from Russian). Oxford: Pergamon, 1961.

◆◇◆◇◆