COUNTING THE "GOOD" SEQUENCES

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For a finite sequence of nonnegative integers, $A = \{a_{1j}\}$, $j = 1, 2, 3, \ldots, n$, define its set of absolute differences by the recursion relation

$$a_{ij} = |a_{i-1,j} - a_{i-1,j+1}|, \text{ for } i + j \leqslant n + 1.$$

We write A along with its set of absolute differences in the natural way indicated in the following table and call the resulting triangular array $T(A)$.

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1   2   4   4   8  10   6
  1   2   0   4   2   4
    1   2   4   2   2
      1   2   2   0
        1   0   2
          1   2
            1
```

If the left "column" of $T(A)$ consists totally of 1's, we say that A is a *good sequence*. There are a great many good sequences of length n, ranging from the "smallest," $\{1, 0, 0, \ldots, 0\}$, to the "largest," $\{1, 2, 4, \ldots, 2^{n-1}\}$. Galbreath conjectured that the sequence $\{p_i - 1\}$, where p_i is the i^{th} prime, is an infinite good sequence (see [1]). A natural question to ask is: *How many good sequences are there of length n?* In this paper, we shall answer this question for small n, and present a heuristic recursion relation.

Let $G(n)$ be the set of good sequences of length n, with $g(n) = \#G(n)$. If $g \in G(n)$, we note that each row of $T(g)$ is a good sequence. This observation, along with the obvious one that any initial subsequence of a good sequence is also good, leads to the following definitions.

For $g \in G(n-1)$, let $e(g) = \#\{g^* \in G(n)$, with g an initial subsequence of $g^*\}$, and $e^*(g) = \#\{g^* \in G(n)$, with g the second row of $T(g^*)\}$. We say that $e(g)$ is the number of ways to extend $T(g)$ to the right, and $e^*(g)$ is the number of ways to extend it upward.

Now, assume $g \in G(n-2)$, and extend $T(g)$ both to the right and upward, as in Figure 1. If we choose c so that $|b - c| = a$, we will have a triangular array that is $T(g^*)$ for some $g^* \in G(n)$. Since c can be chosen in either 1 or 2 ways for a given a and b, based on their relative magnitudes, we have the following equality.

$$g(n) = \sum_{g \in G(n-2)} e(g)e^*(g)\beta(g),$$

where $1 \leqslant \beta(g) \leqslant 2$.

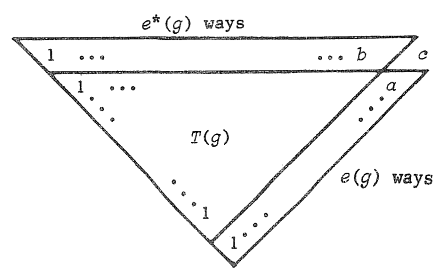

FIGURE 1

The average value of both $e(g)$ and $e^*(g)$ is $g(n - 1)/g(n - 2)$. Also, since a and b are each the last elements of members of $G(n-1)$, we expect $a \leqslant b$ about half the time, and vice versa. In other words, we expect $\beta(g) \sim 3/2$ on average. By replacing $e(g)$, $e^*(g)$, and $\beta(g)$ with these "averages" in the previous sum, we have an "expected" asymptotic recursion relation,

$$g(n) \sim \frac{3}{2}\frac{(g(n - 1))^2}{g(n - 2)}, \text{ as } n \to \infty.$$

To test this relation, $g(n)$ was calculated for $n \leqslant 10$. Its values, along with the values for $\beta(n) = g(n)g(n - 2)/(g(n - 1))^2$, are presented in Table 1.

TABLE 1

n	$g(n)$	$\beta(n)$
1	1	-
2	2	-
3	5	1.250
4	17	1.360
5	82	1.419
6	573	1.449
7	5,839	1.458
8	86,921	1.461
9	1,890,317	1.461
10	60,013,894	1.460

The following questions naturally arise:

Is there a formula for $g(n)$?

Does $\lim_{n \to \infty} g(n)g(n - 2)/(g(n - 1))^2$ exist? If so, what is it?

REFERENCE

1. R. B. Killgrove & K. E. Ralston. "On a Conjecture Concerning the Primes." *Math Tables Aide Comput.* *13* (1959):121-22.

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