A PROPERTY OF NUMBERS EQUIVALENT TO THE GOLDEN MEAN

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We are concerned with finding the convergents $C_j(\alpha) = \dfrac{p_j}{q_j}$, in lowest terms, to a positive real number α that satisfy the inequality,

$$|\alpha - C_j(\alpha)| < \frac{\beta}{\sqrt{5}q_j^2},\ 0 < \beta < 1. \tag{1}$$

From Le Veque [3] or Roberts [4], we have the following theorems.

Hurwitz's theorem states that, if α is irrational and $\beta = 1$, there are infinitely many irreducible rational solutions to (1).

Dirichlet's theorem states that, if $\beta = \sqrt{5}/2$, then all rational solutions to (1) are convergents to α.

Since $1/\sqrt{5} < 1/2$, we note that the expression "irreducible rational solutions" in Hurwitz's theorem may always be replaced by "convergents."

It is readily shown (see [4]) that if $\alpha = \tau = (1 + \sqrt{5})/2$ (the Golden Mean) then there are only finitely many convergents to τ which satisfy (1). In [5], van Ravenstein, Winley, & Tognetti have determined the convergents explicitly.

We now extend [5] by determining the solutions to (1) when α is equivalent to τ, which means the Noble Number α has a simple continued fraction expansion $(a_0; a_1, a_2, \ldots, a_n, 1, 1, 1, \ldots)$ where the terms $a_1, a_2, \ldots, a_n$ are positive integers, $a_n \geqslant 2$ and a_0 is a nonnegative integer.

Using the notation of [5], with C_j replaced by $C_j(\alpha)$, and well-known facts [see Chrystal [1] and Khintchine [2]):

$$\left.\begin{array}{ll}
\text{(i)} & p_j = p_{j-2} + a_j p_{j-1}, \\
 & q_j = q_{j-2} + a_j q_{j-1}, \\
 & \text{for } j \geqslant 0,\ p_{-2} = q_{-1} = 0 \text{ and } q_{-2} = p_{-1} = 1; \\
\text{(ii)} & q_{j+1} > q_j > q_{j-1} > \cdots > q_0 = 1; \\
\text{(iii)} & p_{j-1}q_j - p_j q_{j-1} = (-1)^j; \\
\text{(iv)} & C_j(\tau) = \dfrac{F_{j+1}}{F_j}, \text{ where } F_j \text{ is the } j^{\text{th}} \text{ term of the} \\
 & \text{Fibonacci sequence } \{1, 1, 2, 3, 5, \ldots\}; \\
\text{(v)} & F_j = \dfrac{\tau^{j+1} - (1-\tau)^{j+1}}{\sqrt{5}}.
\end{array}\right\} \tag{2}$$

It follows from (2(1)) that

$$C_j(\alpha) = \frac{p_j}{q_j} = \begin{cases} \dfrac{p_{j-2} + a_j p_{j-1}}{q_{j-2} + a_j q_{j-1}}, & j = 0, 1, 2, \ldots, n \\ \dfrac{F_{j-n} p_n + F_{j-n-1} p_{n-1}}{F_{j-n} q_n + F_{j-n-1} q_{n-1}}, & j = n+1, n+2, \ldots, \end{cases}$$

and

$$\alpha = \lim_{j \to \infty} C_j(\alpha) = \frac{p_{n-1} + \tau p_n}{q_{n-1} + \tau q_n} = C_n(\alpha) + \frac{(-1)^n}{q_n(\tau q_n + q_{n-1})}. \tag{3}$$

Using (2(111)), and (2(iv)) in (3), we see that, for $j \geqslant n + 1$,

$$C_j(\alpha) = \frac{C_{j-n-1}(\tau) p_n + p_{n-1}}{C_{j-n-1}(\tau) q_n + q_{n-1}},$$

$$C_{j-n-1}(\tau) = \frac{F_{j-n}}{F_{j-n-1}},$$

and

$$|\alpha - C_j(\alpha)| = \frac{|\tau - C_{j-n-1}(\tau)|}{(q_{n-1} + q_n\tau)(C_{j-n-1}(\tau) q_n + q_{n-1})} \tag{4}$$

Hence, for $j \geqslant n + 1$, (1) reduces to

$$|\tau - C_{j-n-1}(\tau)| < \frac{\beta(q_{n-1} + q_n\tau)}{\sqrt{5} F_{j-n-1}^2 (C_{j-n-1}(\tau) q_n + q_{n-1})} \tag{5}$$

If $j - n - 1$ is even ($j = n + 1 + 2k$, $k = 0, 1, 2, \ldots$), then using (4) and $\tau^2 = 1 + \tau$ in (5) we seek nonnegative values of k such that

$$(\tau F_{2k} - F_{2k+1})(F_{2k+1} q_n + F_{2k} q_{n-1}) < \frac{\beta}{\sqrt{5}}(q_{n-1} + \tau q_n).$$

Using (2(v)), this reduces to

$$k < \ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1 - \beta)(\tau q_n + q_{n-1})}\right) \Big/ 4 \ln \tau. \tag{6}$$

Now nonnegative values of k in (6) exist only if

$$\ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1 - \beta)(\tau q_n + q_{n-1})}\right) > 0,$$

which means that

$$\beta_u < \beta < 1, \text{ where } \beta_u = \frac{\sqrt{5}}{\tau}\left[\frac{q_n + q_{n-1}}{\tau q_n + q_{n-1}}\right].$$

If $j - n - 1$ is odd ($j = n + 2 + 2k$, $k = 0, 1, 2, \ldots$), then (5) reduces to

$$(F_{2k+2} - \tau F_{2k+1})(F_{2k+2} q_n + F_{2k+1} q_{n-1}) < \frac{\beta}{\sqrt{5}}(q_{n-1} + q_n\tau).$$

Using (2(v)), this further reduces to

$$\tau^{4k+6}(1 - \beta) < \frac{\tau(\tau q_{n-1} - q_n)}{\tau q_n + q_{n-1}}. \tag{7}$$

Since the left side is positive and the right side is negative,

$$\tau - \frac{q_n}{q_{n-1}} < \tau - a_n < 0,$$

there are no nonnegative values of k which satisfy (7).

This completes the solutions to (1) for $j \geqslant n + 1$.

If $j = n$, then from (3) we have

$$|\alpha - C_n(\alpha)| = \frac{1}{q_n(\tau q_n + q_{n-1})},$$

and so (1) becomes

$$\frac{1}{q_n(\tau q_n + q_{n-1})} < \frac{\beta}{\sqrt{5}q_n^2},$$

which means $\beta > \dfrac{\sqrt{5}q_n}{\tau q_n + q_{n-1}}$.

However, since $\tau - (q_n/q_{n-1}) < 0$, we have $q_n > \tau q_{n-1}$, and this gives

$$\beta > \frac{\sqrt{5}q_n}{\tau q_n + q_{n-1}} > 1,$$

which is not possible. Hence, $C_n(\alpha)$ does not satisfy (1).

Consequently, there are no convergents that satisfy (1) if $\beta \leqslant \beta_u$ and $j \geqslant n$.

On the other hand, if $\beta > \beta_u$, then there are $[S] + 1$ convergents that satisfy (1). They are given by

$$C_j(\alpha) = \frac{F_{j-n}p_n + F_{j-n-1}p_{n-1}}{F_{j-n}q_n + F_{j-n-1}q_{n-1}}, \quad j = n+1,\ n+3,\ \ldots,\ n+1+2[S], \tag{8}$$

where

$$S = \ln\left(\frac{q_n - \tau q_{n-1}}{\tau^3(1-\beta)(\tau q_n + q_{n-1})}\right)\bigg/ 4 \ln \tau,$$

and $[S]$ denotes the integer part of S.

We note that if $n = 0$, then $\alpha = (a_0; 1, 1, 1, \ldots)$, $a_0 \geqslant 2$, and the result (8) reduces to that given in [5].

It does not appear to be possible to make a precise statement as to which of the convergents $C_j(\alpha)$ for $j = 0, 1, 2, \ldots, n - 1$ will satisfy (1) without knowing the values of $a_0, a_1, \ldots, a_{n-1}$. However, we have shown that, if $0 < \beta < 1$, then there are only finitely many convergents to α which satisfy (1).

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