ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

<u>H-412</u> *Proposed by Andreas N. Philippou and Frosso S. Makri, University of Patras, Patras, Greece*

Show that

$$\sum_{i=0}^{k-1} \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} = \binom{n}{r}, \; k \geqslant 1, \; 0 \leqslant r \leqslant k - 1 \leqslant n,$$

where the inner summation is over all nonnegative integers $n_1, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - i$ and $n_1 + \cdots + n_k = n - r$.

<u>H-413</u> *Proposed by Gregory Wulczyn, Bucknell University (retired), Lewisburg, PA*

Let m, n be integers. If m and n have the same parity, show that

(1) $(2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(2) $(2m + 1)F_{2n+1} - (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

If m and n have the opposite parity, show that

(3) $(2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod 5$;

(4) $(2m + 1)F_{2n+1} + (2n + 1)F_{2m+1} \equiv 0 \pmod{25}$ if either

(a) $2m + 1$ or $2n + 1$ is a multiple of 5, or
(b) $m \equiv n \equiv 0$ or $m \equiv n \equiv -1 \pmod 5$.

<u>H-414</u> *Proposed by Larry Taylor, Rego Park, NY*

Let j, k, m, and n be integers. Prove that

$$F_{m+j}F_{n+k} = F_{m+k}F_{n+j} - F_{k-j}F_{n-m}(-1)^{m+j}.$$

SOLUTIONS

A Wind from the Past

H-307 *Proposed by Larry Taylor, Briarwood, NY*
(Vol. 17, no. 4, December 1979)

(A) If $p \equiv \pm 1 \pmod{10}$ is prime, $x \equiv \sqrt{5}$ and

$$a \equiv \frac{2(x - 5)}{x + 7} \pmod{p},$$

prove that a, $a + 1$, $a + 2$, $a + 3$, and $a + 4$ have the same quadratic character modulo p if and only if $11 < p \equiv 1$ or $11 \pmod{60}$ and $(-2x/p) = 1$.

(B) If $p \equiv 1 \pmod{60}$, $(2x/p) = 1$, and

$$b \equiv \frac{-2(x + 5)}{7 - x} \pmod{p},$$

then b, $b + 1$, $b + 2$, $b + 3$, and $b + 4$ have the same quadratic character modulo p. Prove that $(11ab/p) = 1$.

Solution by the proposer

(A) Let $f \equiv (x + 1)/2 \pmod{p}$. Then

$$\begin{aligned}
(x + 7)a &\equiv 2x - 10 \equiv -4xf^{-1}\\
(x + 7)(a + 1) &\equiv 3x - 3 \equiv 6f^{-1},\\
(x + 7)(a + 2) &\equiv 4x + 4 \equiv 8f,\\
(x + 7)(a + 3) &\equiv 5x + 11 \equiv 2f^5,\\
(x + 7)(a + 4) &\equiv 6x + 18 \equiv 12f^2 \pmod{p}.
\end{aligned}$$

But $(f^{-1}/p) = (f/p) = (f^5/p)$ and $(4/p) = (f^2/p) = 1$. Therefore,

$$\left(\frac{-4xf^{-1}}{p}\right) = \left(\frac{6f^{-1}}{p}\right) \text{ if and only if } (-2x/p) = (3/p);$$

$$\left(\frac{6f^{-1}}{p}\right) = \left(\frac{8f}{p}\right) \text{ if and only if } (3/p) = 1;$$

$$\left(\frac{8f}{p}\right) = \left(\frac{2f^5}{p}\right) \text{ unconditionally;}$$

$$\left(\frac{2f^5}{p}\right) = \left(\frac{12f^2}{p}\right) \begin{array}{l}\text{if and only if } (6f/p) = 1,\\ \text{if and only if } (3(x + 1)/p) = 1.\end{array}$$

Then, the five consecutive residues have the same quadratic character modulo p if and only if

$$(-2x/p) = ((x + 1)/p) = (3/p) = 1.$$

The following result is given in [1], page 24:

$$\left(\frac{\sqrt{p}}{5}\right) = \left(\frac{-2x(x + 1)}{p}\right).$$

Then $(-2x/p) = ((x + 1)/p)$ if and only if $(\sqrt{p}/5) = 1$. But $(\sqrt{p}/5) = (3/p) = 1$ is equivalent to $p \equiv 1$ or $11 \pmod{60}$. Since

$$(\sqrt{p}/5) = 1 \text{ if } (-2x/p) = ((x + 1)/p) = 1$$

and

$$(\sqrt{p}/5) = 1 \text{ if } (-2x/p) = ((x + 1)/p) = -1,$$

it is necessary to include either $(-2x/p) = 1$ or $((x + 1)/p) = 1$ in the statement of the criterion.

Finally, if $p = 11$ and $(-2x/p) = 1$, then $x \equiv 4$ and $x + 7 \equiv 0 \pmod{11}$, so this result is not valid for $p = 11$.

(B) The second part of this problem should have been stated more generally as follows: If $p \neq 11$ and

$$b \equiv \frac{-2(x + 5)}{7 - x} \pmod{p},$$

prove that $(11ab/p) = 1$.

Then

$$ab \equiv \frac{(2(x - 5))(-2(x + 5))}{(x + 7)(7 - x)} \equiv 20/11 \pmod{p}$$

and the result follows.

Comment: There is a five-term arithmetic progression of Fibonacci-Lucas identities corresponding to this set of five consecutive residues having the same quadratic character modulo p, as follows:

$$-2L_{n-1};\ 3F_{n-1};\ 4F_{n+1};\ F_{n+5};\ 6F_{n+2}.$$

The common difference is $F_n + L_{n+1}$ (i.e., $-2L_{n-1} + F_n + L_{n+1} = 3F_{n-1}$, etc.).

Reference

1. Emma Lehmer. "Criteria for Cubic and Quartic Residuacity." *Mathematika* **5** (1958):20-29.

Somethings Are Constant

H-390 *Proposed by M. Wachtel, Zurich, Switzerland (Vol. 23, no. 3, August 1985)*

For every m,

$2F_{2-m}F_{5+m} + (-1)^m(F_mF_{m+1} + F_{m+2}^2)$ has the unique value 11.

Find a general formula for analogous constant values, which should represent the terms of an infinite sequence.

Prove that no divisor of any of these terms is congruent to 3 or 7 modulo 10.

Solution by Bjorn Poonen, Harvard College, Cambridge, MA

Since $F_n = \dfrac{a^n - b^n}{a - b}$, where $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$, we have:

$$\begin{aligned}
(a - b)^2&[2F_{k-m}F_{k+3+m} + (-1)^{m+k}(F_mF_{m+1} + F_{m+2}^2)] \\
&= 2(a^{k-m} - b^{k-m})(a^{k+3+m} - b^{k+3+m}) + (-1)^{m+k}[(a^m - b^m)(a^{m+1} - b^{m+1}) \\
&\quad + (a^{m+2} - b^{m+2})^2] \\
&= 2(a^{2k+3} + b^{2k+3} - a^{k-m}b^{k+3+m} - a^{k+3+m}b^{k-m}) \\
&\quad + (-1)^{m+k}[a^{2m+1} + b^{2m+1} - a^mb^{m+1} - a^{m+1}b^m \\
&\quad + a^{2m+4} + b^{2m+4} - 2(ab)^{m+2}] \\
&= 2[a^{2k+3} + b^{2k+3} - (ab)^{k-m}(a^{2m+3} + b^{2m+3})] \\
&\quad + (-1)^{k-m}[a^{2m+1} + b^{2m+1} - (ab)^m(a + b) + a^{2m+4} + b^{2m+4} - 2(-1)^{m+2}]
\end{aligned}$$

(continued)

$$\begin{aligned}
&= 2(a^{2k+3} + b^{2k+3}) - 2(-1)^{k-m}(a^{2m+3} + b^{2m+3}) \\
&\quad + (-1)^{k-m}(a^{2m+1} + b^{2m+1}) - (-1)^{k-m}(-1)^m(1) \\
&\quad + (-1)^{k-m}(a^{2m+4} + b^{2m+4}) - 2(-1)^{k-m}(-1)^m \\
&= 2(a^{2k+3} + b^{2k+3}) - (-1)^k - 2(-1)^k + (-1)^{k-m}(a^{2m+4} - 2a^{2m+3} + a^{2m+1}) \\
&\quad + (-1)^{k-m}(b^{2m+4} - 2b^{2m+3} + b^{2m+1}) \\
&= [a^{2k+3} + b^{2k+3} - 4(-1)^k] + [a^{2k+3} + b^{2k+3} + (-1)^k] \\
&\quad + (-1)^{k-m}a^{2m+1}(a-1)(a^2 - a - 1) + (-1)^{k-m}b^{2m+1}(b-1)(b^2 - b - 1) \\
&= [a^{2k+3} + b^{2k+3} - (ab)^k(a^3 + b^3)] + [a^{2k+3} + b^{2k+3} - (ab)^{k+1}(a+b)] \\
&= (a^{k+3} - b^{k+3})(a^k - b^k) + (a^{k+2} - b^{k+2})(a^{k+1} - b^{k+1}) \\
&= (a-b)^2(F_{k+3}F_k + F_{k+2}F_{k+1}).
\end{aligned}$$

Thus,

$$2F_{k-m}F_{k+3+m} + (-1)^{m+k}(F_mF_{m+1} + F_{m+2}^2) = F_{k+3}F_k + F_{k+2}F_{k+1},$$

which yields the result given in the problem when $k = 2$. Now, we wish to show that no divisor of $F_{k+3}F_k + F_{k+2}F_{k+1}$ is congruent to 3 or 7 modulo 10. Let $x = F_k$ and $y = F_{k+1}$. Then

$$F_{k+3}F_k + F_{k+2}F_{k+1} = [(x+y) + y]x + (x+y)y = x^2 + 3xy + y^2.$$

Suppose that $x^2 + 3xy + y^2 \equiv 0 \pmod{p}$ for some prime p. Now, x and y could not both be divisible by p because then all the Fibonacci numbers would be divisible by p. Then, since the discriminant of the quadratic form $x^2 + 3xy + y^2$ is 5, if p is not 2 or 5, we must have $(5/p) = 1$, but by the Law of Quadratic Reciprocity, this is true iff $(p/5) = 1$, which holds iff $p \equiv \pm 1 \pmod{5}$. Now, suppose there were a factor d of $F_{k+3}F_k + F_{k+2}F_{k+1}$ congruent to 3 or 7 modulo 10. Clearly, d has no factors of 2 or 5, so, by the above arguments, d is a product of primes congruent to ± 1 modulo 5. But any product of this sort is itself congruent to ± 1 modulo 5. Thus, d could not be congruent to 3 or 7 modulo 10.

Also solved by P. Bruckman, L. A. G. Dresel, and L. Kuipers.

The Law of Exclusion

H-391 *Proposed by Lawrence Somer, Washington, D.C.*
(Vol. 23, no. 3, August 1985)

For every n, show that no integral divisor of L_{2n} is congruent to 11, 13, 17, or 19 modulo 20. (This problem was suggested by Problem H-364 on p. 313 of the November 1983 issue of *The Fibonacci Quarterly*.)

Solution by L. A. G. Dresel, Reading, England

Let N_0 be the set of integers congruent to 1, 3, 7, or 9 modulo 20, and let N_1 be the set of integers congruent to 11, 13, 17, or 19 modulo 20. Then, since the product of any two integers in N_0 also belongs to N_0, it follows that any integer in N_1 is either a prime or divisible by at least one prime belonging to N_1. Hence, it is sufficient to show that, for all n, L_{2n} is not divisible by any prime belonging to N_1.

For the case of primes congruent to 13 or 17 (mod 20), this has been proved by Paul Bruckman in his solution to H-364, this journal Vol. 23, no. 4 (1985): 283-84.

Thus, there remains the case of primes p congruent to 11 or 19 (mod 20).

For these primes, we have $L_{p-1} \equiv 2 \pmod{p}$ and $\frac{1}{2}(p-1)$ is odd. We also have the identity

$$L_t^2 = L_{2t} + 2(-1)^t,$$

so that putting $t = \frac{1}{2}(p - 1)$, we have

$$L_{\frac{1}{2}(p-1)}^2 \equiv 0 \pmod{p}$$

and, therefore,

$$L_{\frac{1}{2}(p-1)} \equiv 0 \pmod{p}.$$

Then, if e denotes the entry point of p in the Lucas sequence, we have that e divides $\frac{1}{2}(p - 1)$ and, therefore, e is odd. Furthermore, L_k will be divisible by p only when k is an odd multiple of the entry point e, and any such k is also odd.

Hence, L_{2n} is not divisible by any prime congruent to 11 or 19 (mod 20). ∎

Also solved by P. Bruckman, B. Poonen, and the proposer.

Editorial Note: The following problems are as yet unsolved:

H-146, H-148, H-152, H-170, H-179, H-203, H-204, H-211, H-212, H-213, H-214, H-215, H-222, H-260, H-271, H-287, H-300, H-304, H-306, H-307, H-309, H-357, H-365.

LET'S CLEAN UP SOME OF THESE OLDIES!

ADDITIONAL PROBLEM PROPOSALS ARE NEEDED—PITCH IN AND HELP!!

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