SOME PROPERTIES OF THE SEQUENCE $\{W_n(a, b; p, q)\}$

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(*Submitted August 1985*)

1. INTRODUCTION

Elsewhere in this journal [5], the sequence $\{W_n(a, b; p, q)\}$ has been introduced and its basic properties exhibited. Here, we investigate the finite sum of W_k^t (k from 0 to $n - 1$) and the properties of W_{mn}. Notation and content of [5] are assumed, when required.

Particular cases of $\{W_n\}$ are the sequences $\{U_n\}$, $\{V_n\}$, $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$ given by:

$$U_n(p, q) = W_n(1, p; p, q) \tag{1}$$

$$V_n(p, q) = W_n(2, p; p, q) = pU_{n-1}(p, q) - 2qU_{n-2}(p, q) \tag{2}$$

$$H_n(r, s) = W_n(r, r + s; 1, -1) = rF_{n+1} + sF_n \tag{3}$$

$$F_n = W_n(0, 1; 1, -1) = H_n(0, 1) = U_{n-1}(1, -1) \tag{4}$$

$$L_n = W_n(2, 1; 1, -1) = H_n(2, -1) = V_n(1, -1) \tag{5}$$

Historical information about these second-order recurrence sequences can be found in L. Dickson [3]. Of course, $\{F_n\}$ is the famous Fibonacci sequence, $\{L_n\}$ is the Lucas sequence, $\{U_n\}$ and $\{V_n\}$ are generalizations of these, and $\{H_n\}$, discussed in [4], is a different generalization of them, while $\{W_n\}$ is the complete generalization of them. Chief properties of $\{W_n\}$, $\{U_n\}$, $\{V_n\}$, $\{H_n\}$, $\{F_n\}$, and $\{L_n\}$ can be found, for example, in V. E. Hoggatt, Jr. [3], A. F. Horadam [4], [5], [6], D. Jarden [7], E, Lucas [8], K. Subba Rao [9], A. Tagiuri [10], [11], and N. N. Vorobév [12].

Two interesting specializations of (1) and (2) are the Fermat sequences

$$\{U_n(3, 2)\} = \{2^{n+1} - 1\} \quad \text{and} \quad \{V_n(3, 2)\} = \{2^n + 1\}$$

and the Pell sequences

$$\{U_n(2, -1)\} \quad \text{and} \quad \{V_n(2, -1)\}$$

(see [1], [6], [8]).

From (1)-(5), it follows (See [4], [5], [6]) that ($p^2 \neq 4q$),

$$\begin{cases} W_n = \{(b - a\beta)\alpha^n + (a\alpha - b)\beta^n\}/(\alpha - \beta) & (6) \\ U_n = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) & (7) \\ V_n = \alpha^n + \beta^n & (8) \\ H_n = \{(r + s - r\beta_0)\alpha_0^n - (r + s - r\alpha_0)\beta_0^n\}/\sqrt{5} & (9) \\ F_n = (\alpha_0^n - \beta_0^n)/\sqrt{5} & (10) \\ L_n = \alpha_0^n + \beta_0^n & (11) \end{cases}$$

where

$$\alpha = (p + \sqrt{p^2 - 4q})/2,\ \beta = (p - \sqrt{p^2 - 4q})/2$$
$$\alpha_0 = (1 + \sqrt{5})/2, \text{ and } \beta_0 = (1 - \sqrt{5})/2.$$

In the meantime, from [4], [5], and [6], we have:

$$\begin{cases} W_{k+1} = pW_k - qW_{k-1} & (12) \\ W_{k+1}W_{k-1} = W_k^2 + eq^{k-1}, \text{ where } e = abp - a^2q - b^2 & (13) \\ W_{n+k} = W_nV_k - q^kW_{n-k} & (14) \\ W_{m+r}U_{n-r-1} - q^kW_{m+r-k}U_{n-r-k-1} = W_{m+n-k}U_{k-1} & (15) \\ W_{m+r}W_{n-r} - q^kW_{m+r-k}W_{n-r-k} = (bW_{m+n-k} - aqW_{m+n-k-1})U_{k-1} & (16) \end{cases}$$

2. THE FINITE SUM $\sum_{k=0}^{n-1} W_k^t$

Define

$$G_k(m, j) = \sum_{i=0}^{m} \binom{m}{i} W_{k+1}^{m+j-i+1}(qW_{k-1})^{j+i+1}; \tag{17}$$

we have

Lemma 1: $G_k(m, j) = q^{j+1}(pW_k)^m(W_k^2 + eq^{k-1})^{j+1}$ (18)

$$= p^m \left\{ \sum_{i=0}^{j+1} \binom{j+1}{i} e^{j-i+1} q^{k(j-i+1)+i} W_k^{m+2i} \right\}, \tag{19}$$

where $e = abp - a^2q - b^2$.

Proof: $G_k(m, j) = \sum_{i=0}^{m} \binom{m}{i} W_{k+1}^{m+j-i+1}(qW_{k-1})^{j+i+1}$, by (17)

$$= (qW_{k+1}W_{k-1})^{j+1} \left\{ \sum_{i=0}^{m} \binom{m}{i} W_{k+1}^{m-i}(qW_{k-1})^i \right\}$$

$= (qW_{k+1}W_{k-1})^{j+1}(W_{k+1} + qW_{k-1})^m$, by the binomial theorem

$= q^{j+1}(W_k^2 + eq^{k-1})^{j+1}(W_{k+1} + qW_{k-1})^n$, by (13)

$= q^{j+1}(W_k^2 + eq^{k-1})^{j+1}(pW_k)^m$, by (12)

$= q^{j+1}(pW_k)^m \left\{ \sum_{i=0}^{j+1} \binom{j+1}{i} W_k^{2i}(eq^{k-1})^{j-i+1} \right\}$, by the binomial theorem

$$= p^m \left\{ \sum_{i=0}^{j+1} \binom{j+1}{i} e^{j-i+1} q^{k(j-i+1)+i} W_k^{m+2i} \right\}.$$

Consider $a_j(t)$ satisfying the following recurrence,

$$a_{j+1}(t + 2) = a_{j+1}(t + 1) + a_j(t), \tag{20}$$

subject to the initial conditions $a_0(t) = 1$ for $t \geqslant 1$, $a_j(2j) = 2$ for $j \geqslant 0$, with $a_j(t) = 0$ for $j < 0$ and $j > [t/2]$. It is easy to prove directly from (20) that

$$a_j(t) = \binom{t-j}{j} + \binom{t-j-1}{j-1}. \tag{21}$$

The first few value of $a_j(t)$ are shown in Table 1.

Table 1. The Values of $a_j(t)$

$j \backslash t$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	0	0	0	2	5	9	14	20	27	35	44	54	65	77	90
3	0	0	0	0	0	0	2	7	16	30	50	77	112	156	210	275
4	0	0	0	0	0	0	0	0	2	9	25	55	105	182	294	450
5	0	0	0	0	0	0	0	0	0	0	2	11	36	91	196	378
6	0	0	0	0	0	0	0	0	0	0	0	0	2	13	49	140
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	15

Lemma 2: $$\sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i = \sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)G_k(t-2j, j-1). \tag{22}$$

Proof: $\displaystyle\sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i = \sum_{i=0}^{t-2}\binom{t}{i+1}W_{k+1}^{t-i-1}(qW_{k-1})^{i+1}$, by a dummy variable

$$= t\sum_{i=0}^{t-2}\binom{t-2}{i}W_{k+1}^{t-i-1}(qW_{k-1})^{i+1} - \frac{t}{2}\binom{t-3}{1}\sum_{i=0}^{t-4}\binom{t-4}{i}W_{k+1}^{t-i-2}(qW_{k-1})^{i+2}$$

$$+ \frac{t}{3}\binom{t-4}{2}\sum_{i=0}^{t-6}\binom{t-6}{i}W_{k+1}^{t-i-3}(qW_{k-1})^{i+3} - \cdots, \text{ by expansion}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)\left\{\sum_{i=0}^{t-2j}\binom{t-2j}{i}W_{k+1}^{t-j-i}(qW_{k-1})^{j+i}\right\}, \text{ by summation}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)G_k(t-2j, j-1), \text{ by (17).}$$

Consider $A(j, t; p, q) \equiv A(j, t)$ satisfying the following recurrence,

$$A(j+1, t+2) = pA(j+1, t+1) - qA(j+1, t) + A(j, t) \tag{23}$$

subject to the initial conditions $A(j, 2j) = 2$ for $j \geqslant 0$, $A(0, 1) = p$, with $A(j, t) = 0$ for $j < 0$ and $j > [t/2]$. It is easy to prove directly from (23) that

$$A(j, t) = p^{t-2j}\left\{\sum_{i=0}^{[t/2]-j}\binom{i+j}{j}(-p^{-2}q)^i a_{i+j}(t)\right\}. \tag{24}$$

The first few values of $A(j, t)$ are shown in Table 2. Note that

$$A(j, t) = \frac{(-1)^j}{j!}V_t^{(j)}, \text{ where } V_t^{(j)} = \frac{\partial^j V_t}{\partial q^j}.$$

Table 2. The Values of $A(j, t)$

t \ j	0	1	2	3	4
0	2	0	0	0	0
1	p	0	0	0	0
2	$p^2 - 2q$	2	0	0	0
3	$p^3 - 3pq$	$3p$	0	0	0
4	$p^4 - 4p^2q + 2q^2$	$4p^2 - 4q$	2	0	0
5	$p^5 - 5p^3q + 5pq^2$	$5p^3 - 10pq$	$5q$	0	0
6	$p^6 - 6p^4q + 9p^2q^2 - 2q^3$	$6p^4 - 18p^2q + 6q^2$	$9p^2 - 6q$	2	0
7	$p^7 - 7p^5q + 14p^3q^2 - 7pq^3$	$7p^5 - 28p^3q + 21pq^2$	$14p^3 - 21pq$	$7p$	0

Now, define

$$L_W(r, t) = \sum_{k=0}^{n-1} q^{kr} W_k^t \tag{25}$$

and

$$W(t) = \sum_{k=0}^{n-1} W_k^t = L_W(0, t), \tag{26}$$

where r and t are nonnegative integers; then we have the following lemmas and theorem.

Lemma 3:
$$\sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) p^{t-2j} \left\{ \sum_{i=1}^{j} \binom{j}{i} e^i q^{j-i} L_W(r + i, t - 2i) \right\} \tag{27}$$
$$= -\sum_{j=1}^{[t/2]} (-e)^j A(j, t) L_W(r + j, t - 2j).$$

Proof:
$$\sum_{j=1}^{[t/2]} (-1)^{j+1} a_j(t) p^{t-2j} \left\{ \sum_{i=1}^{j} \binom{j}{i} e^i q^{j-i} L_W(r + i, t - 2i) \right\}$$
$$= a_1(t) p^{t-2} e L_W(r + 1, t - 2) - a_2(t) p^{t-4} \left\{ \sum_{i=1}^{2} \binom{2}{i} e^i q^{2-i} L_W(r + i, t - 2i) \right\}$$
$$+ a_3(t) p^{t-6} \left\{ \sum_{i=1}^{3} \binom{3}{i} e^i q^{3-i} L_W(r + i, t - 2i) \right\} - \cdots, \text{ by expansion}$$
$$= eA(1, t) L\ (r + 1, t - 2) - e^2 A(2, t) L_W(r + 2, t - 4)$$
$$+ e^3 A(3, t) L_W(r + 3, t - 6) - \cdots, \text{ by collecting terms in } L_W(r + i, t - 2i) \text{ for all positive integers } i$$
$$= -\sum_{j=1}^{[t/2]} (-e)^j A(j, t) L_W(r + j, t - 2j), \text{ by summation.}$$

Lemma 4:
$$\sum_{k=0}^{n-1} q^{kr} G_k(t - 2j, j - 1) = p^{t-2j} \left\{ \sum_{i=0}^{j} \binom{j}{i} e^i q^{j-i} L_W(r + i, t - 2i) \right\}. \tag{28}$$

Proof: $\sum_{k=0}^{n-1} q^{kr} G_k(t - 2j,\ j - 1) = \sum_{k=0}^{n-1} q^{kr}\{q^j (pW_k)^{t-2j} (W_k^2 + eq^{k-1})^j\}$, by (18)

$$= \sum_{k=0}^{n-1} q^{kr+j}(pW_k)^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i}W^{2j-2i}(eq^{k-1})^i\right\},\ \text{by the binomial theorem}$$

$$= p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i}e^i q^{j-i}\left\{\sum_{k=0}^{n-1}q^{k(r+i)}W_k^{t-2i}\right\}\right\}$$

$$= p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i}e^i q^{j-i}L_W(r + i,\ t - 2i)\right\},\ \text{by (25).}$$

Consider $B(t;\ p,\ q) \equiv B(t)$ satisfying the following recurrence,

$$B(t + 2) = pB(t + 1) - qB(t) + a_0(t)p^t q, \tag{29}$$

subject to the initial conditions $B(0) = B(1) = 0$.

Let $C(t) = B(t) - a_0(t)p^t$; then $C(t)$ satisfies the following recurrence,

$$C(t + 2) = pC(t + 1) - qC(t) \text{ with } C(0) = -2,\ C(1) = -p, \tag{30}$$

i.e.,

$$C(t) = -p^t\left\{\sum_{j=0}^{[t/2]}(-p^{-2}q)^j a_j(t)\right\}, \tag{31}$$

$$B(t) = -p^t\left\{\sum_{j=1}^{[t/2]}(-p^{-2}q)^j a_j(t)\right\}. \tag{32}$$

Table 3. The Values of $B(t)$ and $C(t)$

t	0	1	2	3	4	5
$B(t)$	0	0	$2q$	$3pq$	$4p^2q - 2q^2$	$5p^3q - 5pq^2$
$C(t)$	-2	$-p$	$-p^2 + 2q$	$-p^3 + 3pq$	$-p^4 + 4p^2q - 2q^2$	$-p^5 + 5p^3q - 5pq^2$

Lemma 5: $\sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\}$

$$= B(t)L_W(r,\ t) - \sum_{j=1}^{[t/2]}(-e)^j A(j,\ t)L_W(r + j,\ t - 2j). \tag{33}$$

Proof: $\sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\}$

$$= \sum_{k=0}^{n-1} q^{kr}\left\{\sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)G_k(t - 2j,\ j - 1)\right\},\ \text{by (22)}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)\left\{\sum_{k=0}^{n-1}q^{kr}G_k(t - 2j,\ j - 1)\right\}$$

$$= \sum_{j=1}^{[t/2]}(-1)^{j+1}a_j(t)p^{t-2j}\left\{\sum_{i=0}^{j}\binom{j}{i}e^i q^{j-i}L_W(r + i,\ t - 2i)\right\},\ \text{by (28)}$$

(continued)

$$= B(t)L_W(r, t) - \sum_{j=1}^{[t/2]}(-e)^j A(j, t)L_W(r + j, t - 2j), \text{ by (27) and (32).}$$

Theorem 1: $L_W(r, t)$ satisfies the following recursion,

$$\{1 + q^{2r+t} - a_0(t)p^t q^r + q^r B(t)\}L_W(r, t)$$
$$= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)$$
$$+ q^r\left\{\sum_{j=1}^{[t/2]}(-e)^j A(j, t)L_W(r + j, t - 2j)\right\}, \tag{34}$$

for $t \geqslant 1$ or ($t = 0$ and $r \geqslant 1$).

Proof: (1) When $t = 0$ and $r \geqslant 1$:

$$L_W(r, 0) = \sum_{k=0}^{n-1} q^{kr}, \text{ from (25).}$$

Hence, $L_W(r, 0)$ satisfies (34).

(2) When $t \geqslant 1$: Since

$$p^t L_W(r, t) = \sum_{k=0}^{n-1} q^{kr}(pW_k)^t, \text{ by (25)}$$
$$= \sum_{k=0}^{n-1} q^{kr}(W_{k+1} + qW_{k-1})^t, \text{ by (12)}$$
$$= \sum_{k=0}^{n-1} q^{kr}\left\{\sum_{i=0}^{t}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\}, \text{ by the binomial theorem}$$
$$= \sum_{k=0}^{n-1} q^{kr}\left\{W_{k+1}^t + q^t W_{k-1}^t + \sum_{i=1}^{t-1}\binom{t}{i}W_{k+1}^{t-i}(qW_{k-1})^i\right\}, \text{ by expansion}$$
$$= \{q^{-r}L_W(r, t) + q^{(n-1)r}W_n^t - q^{-r}W_0^t\} + q^t\{q^r L_W(r, t) - q^{nr}W_{n-1}^t + W_{-1}^t\}$$
$$+ B(t)L_W(r, t) - \sum_{j=1}^{[t/2]}(-e)^j A(j, t)L_W(r + j, t - 2j), \text{ by (33),}$$

we have

$$\{q^{-r} + q^{r+t} - p^t + B(t)\}L_W(r, t)$$
$$= q^{(n-1)r}(q^{r+t}W_{n-1}^t - W_n^t) - q^{-r}(q^{r+t}W_{-1}^t - W_0^t)$$
$$+ \sum_{j=1}^{[t/2]}(-e)^j A(j, t)L_W(r + j, t - 2j).$$

Hence,

$$\{1 + q^{2r+t} - p^t q^r + q^r B(t)\}L_W(r, t)$$
$$= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)$$
$$+ q^r\left\{\sum_{j=1}^{[t/2]}(-e)^j A(j, t)L_W(r + j, t - 2j)\right\}.$$

This completes the proof of Theorem 1, since $a_0(t) = 1$ for $t \geqslant 1$.

Setting $t = 0$, 1, 2, and 3 in Theorem 1, we have the following four corollaries.

Corollary 1: $(1 - q^r)L_W(r, 0) = 1 - q^{nr}$, for all $r \geqslant 1$ [cf. (25)].

Proof: Setting $t = 0$ in Theorem 1, we have

$$(1 + q^{2r} - a_0(0)q^r + q^rB(0))L_W(r, 0) = q^{nr}(q^r - 1) - (q^r - 1),$$

i.e.,

$$(1 - q^r)L_W(r, 0) = 1 - q^{nr}, \text{ since } a_0(0) = 2.$$

See also Proof (1) of Theorem 1.

Corollary 2: $(1 + q^{2r+1} - pq^r)L_W(r, 1) = q^{nr}(q^{r+1}W_{n-1} - W_n) - (q^{r+1}W_{-1} - W_0)$.

Proof: Setting $t = 1$ in Theorem 1, we have

$$(1 + q^{2r+1} - a_0(1)pq^r + q^rB(1))L_W(r, 1)$$
$$= q^{nr}(q^{r+1}W_{n-1} - W_n) - (q^{r+1}W_{-1} - W_0),$$

completing the proof of Corollary 2.

Corollary 3: $(1 + q^{2r+2} - p^2q^r + 2q^{r+1})L_W(r, 2)$

$$= q^{nr}(q^{r+2}W_{n-1} - W_n^2) - (q^{r+2}W_{-1}^2 - W_0^2) - 2eq^rL_W(r + 1, 0).$$

Proof: Setting $t = 2$ in Theorem 1, we have

$$(1 + q^{2r+2} - a_0(2)p^2q^r + q^rB(2))L_W(r, 2)$$
$$= q^{nr}(q^{r+2}W_{n-1}^2 - W_n^2) - (q^{r+2}W_{-1}^2 - W_0^2) - eq^rA(1, 2)L_W(r + 1, 0),$$

completing the proof of Corollary 3.

Corollary 4: $(1 + q^{2r+3} - p^3q^r + 3pq^{r+1})L_W(r, 3)$

$$= q^{nr}(q^{r+3}W_{n-1}^3 - W_n^3) - (q^{r+3}W_{-1}^3 - W_0^3) - 3epq^rL_W(r + 1, 1).$$

Proof: Setting $t = 3$ in Theorem 1, we have

$$(1 + q^{2r+3} - a_0(3)p^3q^r + q^rB(3))L_W(r, 3)$$
$$= q^{nr}(q^{r+3}W_{n-1}^3 - W_n^3) - (q^{r+3}W_{-1}^3 - W_0^3) - eq^rA(1, 3)L_W(r + 1, 1),$$

completing the proof of Corollary 4.

Since $C(t) = B(t) - a_0(t)p^t$, we have

Theorem 1′: $L_W(r, t)$ satisfies the following recursion,

$$\{1 + q^{2r+t} + q^rC(t)\}L_W(r, t)$$
$$= q^{nr}(q^{r+t}W_{n-1}^t - W_n^t) - (q^{r+t}W_{-1}^t - W_0^t)$$
$$+ q^r\left\{\sum_{j=1}^{[t/2]}(-e)^jA(j, t)L_W(r + j, t - 2j)\right\}, \tag{35}$$

for $t \geqslant 1$ or ($t = 0$ and $r \geqslant 1$).

Setting $r = 0$ in Theorem 1′, we have

Theorem 2: $W(t)$ satisfies the following recursion,

$$\{1 + q^t + C(t)\}W(t)$$
$$= (q^t W^t_{n-1} - W^t_n) - (q^t W^t_{-1} - W^t_0) + \sum_{j=1}^{[t/2]} (-e)^j A(j, t) L_W(j, t - 2j), \quad (36)$$

for $t \geqslant 1$.

Now, we have the following five formulas about $W(t)$ for t, respectively, 1 to 5:

$$(1 + q - p)W(1) = (qW_{n-1} - W_n) - (qW_{-1} - W_0); \quad (37)$$
$$(1 + q^2 - p^2 + 2q)W(2) = (q^2W^2_{n-1} - W^2_n) - (q^2W^2_{-1} - W^2_0) - 2eL_W(1, 0); \quad (38)$$
$$(1 + q^3 - p^3 + 3pq)W(3) = (q^3W^3_{n-1} - W^3_n) - (q^3W^3_{-1} - W^3_0) - 3epL_W(1, 1); \quad (39)$$
$$(1 + q^4 - p^4 + 4p^2q - 2q^2)W(4)$$
$$= (q^4W^4_{n-1} - W^4_n) - (q^4W^4_{-1} - W^4_0) - 4e(p^2 - q)L_W(1, 2) + 2e^2L_W(2, 0); \quad (40)$$
$$(1 + q^5 - p^5 + 5p^3q - 5pq^2)W(5)$$
$$= (q^5W^5_{n-1} - W^5_n) - (q^5W^5_{-1} - W^5_0) - 5ep(p^2 - 2q)L_W(1, 3) + 5e^2pL_W(2, 1). \quad (41)$$

We note that (37) is the equivalent form of (3.5) in [5], (38) is the simple form of (4.16) in [5], and (39) is the simple form of (4.28), misprinted, in [5].

Finally, we consider the corresponding special cases of $W(t)$:

(1) When $a = r$, $b = r + s$, $p = 1$, and $q = -1$, then $H(t) = \sum_{k=0}^{n-1} H^t_k(r, s)$ has the following properties:

$H(1) = H_n + H_{n-1} - r - s = H_{n+1} - r - s$, by (37);

$H(2) = H^2_n - H^2_{n-1} - r^2 + s^2 + (1 - (-1)^n)(r^2 - rs - s^2)$, by (38) and Cor. 1;

$4H(3) = H^3_n + H^3_{n-1} - r^3 - s^3 + 3(r^2 - rs - s^2)\{(-1)^{n+1}H_{n-2} + r - s\}$, by (39) and Cor. 2;

$5H(4) = H^4_n - H^4_{n-1} - r^4 + s^4 + 6n(r^2 - rs - s^2)^2/5 + 8(r^2 - rs - s^2)\{(-1)^{n+1}(H^2_n + H^2_{n-1}) + r^2 + s^2\}/5$, by (40) and Cors. 1, 3;

$11H(5) = H^5_n + H^5_{n-1} - r^5 - s^5 + 25(r^2 - rs - s^2)^2(H_{n+1} - r - s)/4 + 15(r^2 - rs - s^2)\{(-1)^{n+1}(H^3_n - H^3_{n-1}) + r^3 - s^3\}/4$, by (41) and Cors. 2, 4.

(2) When $a = 0$, $b = p = 1$, and $q = -1$, then $F(t) = \sum_{k=0}^{n-1} F^t_k$ has the following properties:

$F(1) = F_{n+1} - 1$

$F(2) = F^2_n - F^2_{n-1} + (-1)^n = (F_{2n} - F^2_n)/2$

$4F(3) = F^3_n + F^3_{n-1} + 3(-1)^nF_{n-2} + 2$

$5F(4) = F^4_n - F^4_{n-1} + 8(-1)^n(F^2_n + F^2_{n-1})/5 + 6n/5 - 3/5$

$11F(5) = F^5_n + F^5_{n-1} + 15(-1)^n(F^3_n - F^3_{n-1})/4 + 25F_{n+1}/4 - 7/2$

(3) When $a = 2$, $b = p = 1$, and $q = -1$, then $L(t) = \sum_{k=0}^{n-1} L^t_k$ has the following properties:

$L(1) = L_{n+1} - 1$

$$L(2) = L_n^2 - L_{n-1}^2 + 5(-1)^{n+1} + 2$$
$$4L(3) = L_n^3 + L_{n-1}^3 + 15(-1)^{n+1}L_{n-2} + 38$$
$$5L(4) = L_n^4 - L_{n-1}^4 + 8(-1)^{n+1}(L_n^2 + L_{n-1}^2) + 30n + 25$$
$$11L(5) = L_n^5 + L_{n-1}^5 + 75(-1)^{n+1}(L_n^3 - L_{n-1}^3)/4 + 625L_{n+1}/4 - 37/2$$

3. THE PROPERTIES OF W_{mn}

Define
$$\tilde{L}_m(q) \equiv \tilde{L}_m = \sum_{k=0}^{[(m-1)/2]} \binom{m-k-1}{k}(-q^n)^k V_n^{m-2k-1}, \quad \text{with } \tilde{L}_0 = 0,$$
where m and n are nonnegative integers. Then we obtain the following lemma.

Lemma 6: $\tilde{L}_m$ satisfies the following recursion,
$$\tilde{L}_{m+2} = V_n\tilde{L}_{m+1} - q^n\tilde{L}_m, \text{ with } \tilde{L}_0 = 0 \text{ and } \tilde{L}_1 = 1.$$

Using Lemma 6 and mathematical induction, we have

Theorem 3: $W_{mn} = \tilde{L}_m W_n - aq^n\tilde{L}_{m-1}$.

Proof: For $m = 1$, we have $W_n = \tilde{L}_1 W_n - aq^n\tilde{L}_0$ from the definition and from the formula. Similarly, the theorem is true if $m = 2$. We now show that the formula for $m + 1$ follows from the formula for m and $m - 1$.

$$\begin{aligned} W_{(m+1)n} &= V_n W_{mn} - q^n W_{(m-1)n}, \text{ by (14)} \\ &= V_n(\tilde{L}_m W_n - aq^n\tilde{L}_{m-1}) - q^n(\tilde{L}_{m-1}W_n - aq^n\tilde{L}_{m-2}) \\ &= (V_n\tilde{L}_m - q^n\tilde{L}_{m-1})W_n - aq^n(V_n\tilde{L}_{m-1} - q^n\tilde{L}_{m-2}) \\ &= \tilde{L}_{m+1}W_n - aq^n\tilde{L}_m, \text{ by Lemma 6,} \end{aligned}$$

completing the proof.

In particular, we have the following six corollaries.

Corollary 5: $U_{mn-1} = \tilde{L}_m U_{n-1}$, i.e., $U_{n-1} | U_{mn-1}$.

Corollary 6:
$$U_{mn} = \tilde{L}_m U_n - q^n\tilde{L}_{m-1} = \sum_{k=0}^{\infty} (-q^n)^k V_n^{m-2k-2}\left\{\binom{m-k-1}{k}U_nV_n - \binom{m-k-2}{k}q^n\right\}.$$

Corollary 7:
$$V_{mn} = \tilde{L}_m V_n - 2q^n\tilde{L}_{m-1} = V_n^m + \sum_{k=1}^{\infty}(-q^n)^k V_n^{m-2k}a_k(m) = \sum_{k=0}^{\infty}(-q^n)^k V_n^{m-2k-2}\left\{\binom{m-k-1}{k}V_n^2 - 2\binom{m-k-2}{k}q^n\right\}.$$

That is to say, $V_n | V_{mn}$ if m is odd.

Corollary 8:
$$H_{mn}(r, s) = \tilde{L}_m(-1)H_n(r, s) - r(-1)^n\tilde{L}_{m-1}(-1) = \sum_{k=0}^{\infty}(-1)^{(n+1)k}L_n^{m-2k-2}\left\{\binom{m-k-1}{k}L_nH_n(r, s) + r(-1)^{n+1}\binom{m-k-2}{k}\right\}.$$

Corollary 9: $F_{mn} = \tilde{L}_m(-1)F_n = \sum_{k=0}^{\infty}(-1)^{(n+1)k}\binom{m-k-1}{k}L_n^{m-2k-1}F_n$, i.e., $F_n | F_{mn}$.

Corollary 10: $L_{mn} = \tilde{L}_m(-1)L_n - 2(-1)^n\tilde{L}_{m-1}(-1)$

$$= L_n^m + \sum_{k=1}^{\infty}(-1)^{(n+1)k}L_n^{m-2k}a_k(m)$$

$$= \sum_{k=0}^{\infty}(-1)^{(n+1)k}L_n^{m-2k-2}\left\{\binom{m-k-1}{k}L_n^2 + 2(-1)^{n+1}\binom{m-k-2}{k}\right\}.$$

That is to say, $L_n | L_{mn}$ if m is odd.

Example 1: Setting $m = 2$, we have the following seven properties:

$W_{2n} = V_nW_n - aq^n$

$U_{2n-1} = V_nU_{n-1}$ (see [5]; [8])

$U_{2n} = V_nU_n - q^n$

$V_{2n} = V_n^2 - 2q^n$ (see [5]; [8])

$H_{2n}(r, s) = L_nH_n(r, s) - r(-1)^n$

$F_{2n} = L_nF_n$

$L_{2n} = L_n^2 - 2(-1)^n$

Example 2: Setting $m = 3$, we obtain the following seven properties:

$W_{3n} = (V_n^2 - q^n)W_n - aq^nV_n$

$U_{3n-1} = (V_n^2 - q^n)U_{n-1}$ (see [5]; [8])

$U_{3n} = (V_n^2 - q^n)U_n - q^nV_n$

$V_{3n} = (V_n^2 - 3q^n)V_n$ (see [5]; [8])

$H_{3n}(r, s) = (L_n^2 - (-1)^n)H_n(r, s) - r(-1)^nL_n$

$F_{3n} = (L_n^2 - (-1)^n)F_n$

$L_{3n} = (L_n^2 - 3(-1)^n)L_n$

Example 3: Setting $m = 4$, we have the following seven properties:

$W_{4n} = (V_n^2 - 2q^n)V_nW_n - aq^n(V_n^2 - q^n)$

$U_{4n-1} = (V_n^2 - 2q^n)V_nU_{n-1}$

$U_{4n} = (V_n^2 - 2q^n)V_nU_n - q^n(V_n^2 - q^n)$

$V_{4n} = V_n^4 - 4q^nV_n^2 + 2q^{2n}$

$H_{4n}(r, s) = (L_n^2 - 2(-1)^n)L_nH_n(r, s) - r(-1)^nL_n^2 + r$

$F_{4n} = (L_n^2 - 2(-1)^n)L_nF_n = (L_n^2 - 2(-1)^n)F_{2n}$

$L_{4n} = L_n^4 - 4(-1)^nL_n^2 + 2$

4. THE POWER EXPANSION OF W_n

Since
$$\begin{cases} W_n(1,\, 0;\, p,\, q) = \displaystyle\sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} p^{n-2k}(-q)^k \\ W_n(0,\, 1;\, p,\, q) = \displaystyle\sum_{k=1}^{[(n+1)/2]} \binom{n-k}{k-1} p^{n-2k+1}(-q)^{k-1}, \end{cases}$$
we have
$$W_n(a,\, b;\, p,\, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{bp\binom{n-k}{k-1} - aq\binom{n-k-1}{k-1}\right\}.$$

Now, we consider the special cases of $W_n(a,\, b;\, p,\, q)$:

$$U_n(p,\, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{p^2\binom{n-k}{k-1} - q\binom{n-k-1}{k-1}\right\}$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} p^{n-2j} q^j$$

$$V_n(p,\, q) = \sum_{k=1}^{\infty} p^{n-2k}(-q)^{k-1}\left\{p^2\binom{n-k}{k-1} - 2q\binom{n-k-1}{k-1}\right\}$$

$$H_n(r,\, s) = \sum_{k=0}^{\infty}\left\{r\binom{n-k}{k} + s\binom{n-k-1}{k}\right\} = rF_{n+1} + sF_n$$

$$F_n = \sum_{k=0}^{\infty}\binom{n-k-1}{k}$$

$$L_n = \sum_{k=0}^{\infty}\left\{2\binom{n-k}{k} - \binom{n-k-1}{k}\right\} = 2F_{n+1} - F_n$$

Remark: $W_{mn+k} = \displaystyle\sum_{i=0}^{m}\binom{m}{i} U_{n-1}^{i}(-qU_{n-2})^{m-i} W_{k+i}.$

ACKNOWLEDGMENT

We would like to thank Professor Gou-Sheng Yang for introducing us to this topic. We also appreciate the helpful comments of Professor Horng-Jinh Chang, and the thorough discussions and valuable suggestions of the referee.

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