ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-415 *Proposed by Larry Taylor, Rego Park, N.Y.*

Let n and w be integers with w odd. From the following Fibonacci-Lucas identity (Elementary Problem B-464, *The Fibonacci Quarterly*, December 1981, p. 466), derive another Fibonacci-Lucas identity using the method given in Problem 1:

$$F_{n+2w}F_{n+w} - 2L_wF_{n+w}F_{n-w} - F_{n-w}F_{n-2w} = (L_{3w} - 2L_w)F_n^2.$$

H-416 *Proposed by Gregory Wulczyn, Bucknell University (Ret.), Lewisburg, PA*

(1) If $\left(\frac{p}{5}\right) = 1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv 1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv 1 \pmod{p}. \end{cases}$

(2) If $\left(\frac{p}{5}\right) = -1$, show that $\begin{cases} .5(L_{p-1} + F_{p-1}) \equiv -1 \pmod{p}, \\ .5(L_{p+1} - F_{p+1}) \equiv -1 \pmod{p}. \end{cases}$

H-417 *Proposed by Piero Filipponi, Rome, Italy*

Let $G(n, m)$ denote the geometric mean taken over m consecutive Fibonacci numbers of which the smallest is F_n. It can be readily proved that

$$G(n, 2k+1) \qquad (k = 1, 2, \ldots)$$

is not integral and is asymptotic to F_{n+k} (as n tends to infinity).

Show that if n is odd (even), then $G(n, 2k+1)$ is greater (smaller) than F_{n+k}, except for the case $k = 2$, where $G(n, 5) < F_{n+2}$ for every n.

SOLUTIONS

Bracket Some Sums

H-392 *Proposed by Piero Filipponi, Rome, Italy [Vol. 23(4), Nov. 1985]*

It is known [1], [2], [3], [4] that every positive integer n can be represented uniquely as a finite sum of *F-addends* (distinct nonconsecutive Fibonacci numbers). Denoting by $f(n)$ the number of F-addends the sum of which represents the integer n and denoting by $[x]$ the greatest integer not exceeding x, prove that:

(i) $f([F_k/2]) = [k/3], \quad (k = 3, 4, \ldots);$

(ii) $$f([F_k/3]) = \begin{cases} [k/4] + 1, & \text{for } [k/4] \equiv 1 \pmod 2 \text{ and } k \equiv 3 \pmod 4 \\ [k/4], & \text{otherwise.} \end{cases} \quad (k = 4, 5, \ldots)$$

Find (if any) a closed expression for $f([F_k/p])$ with p a prime and k such that $F_k \equiv 0 \pmod p$.

References

1. J. L. Brown, Jr. "Zeckendorf's Theorem and Some Applications." *The Fibonacci Quarterly* 2, no. 4 (1964):163-168.
2. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* 3, no. 1 (1965):1-8.
3. D. E. Daykin. "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers." *J. London Math. Soc.* 35 (1960):143-160.
4. D. A. Klarner. "Partitions of N into Distinct Fibonacci Numbers." *The Fibonacci Quarterly* 6, no. 4 (1968):235-244.

Solution (partial) by the proposer

Proof (i): Let us put $k = 3h + v \quad (v = 0, 1, 2; h = 1, 2, \ldots)$. On the basis of the equalities

$$[F_{3h+v}/2] = \begin{cases} F_{3h}/2, & \text{for } v = 0 \\ (F_{3h+v} - 1)/2, & \text{for } v = 1, 2 \end{cases}$$

the relations

$$[F_{3h+v}/2] = \sum_{i=1}^{h} F_{3i+v-2} \quad (v = 0, 1, 2)$$

can be proven by induction on h. Therefore $[F_k/2]$ can be represented as a sum of $h = [k/3]$ F-addends.

Proof (ii): Let us put $k = 4h + v \quad (v = 0, 1, 2, 3; h = 1, 2, \ldots)$. By virtue of the identity

$$F_{t+s} = F_{t+1}F_s + F_tF_{s-1} \tag{1}$$

and of the congruence

$$F_{4h} \equiv 0 \pmod 3, \tag{2}$$

the congruences

$$F_{4h+1} \equiv \begin{cases} 1 \pmod 3, & \text{for } h \text{ even,} \\ 2 \pmod 3, & \text{for } h \text{ odd,} \end{cases} \tag{3}$$

can be readily proven by induction on h. From (1) and (2), we can write:

$$[F_{4h+v}/3] = \begin{cases} F_{4h}/3, & \text{for } v = 0, \\ [F_{4h+1}/3], & \text{for } v = 1, \\ F_{4h}/3 + [F_{4h+1}/3], & \text{for } v = 2, \\ F_{4h}/3 + [2F_{4h+1}/3], & \text{for } v = 3; \end{cases} \tag{4}$$

therefore, from (3) and (4), we obtain:

$$[F_{4h}/3] = F_{4h}/3, \forall h; \tag{5}$$

$$[F_{4h+1}/3] = \begin{cases}(F_{4h+1} - 1)/3, \text{ for } h \text{ even,}\\(F_{4h+1} - 2)/3, \text{ for } h \text{ odd;}\end{cases} \tag{5'}$$

$$[F_{4h+2}/3] = \begin{cases}(F_{4h+2} - 1)/3, \text{ for } h \text{ even,}\\(F_{4h+2} - 2)/3, \text{ for } h \text{ odd;}\end{cases} \tag{5''}$$

$$[F_{4h+3}/3] = \begin{cases}(F_{4h+3} - 2)/3, \text{ for } h \text{ even,}\\(F_{4h+3} - 1)/3, \text{ for } h \text{ odd.}\end{cases} \tag{5'''}$$

From (5), (5′), (5″), (5‴), and on the basis of (1) and of the identity

$$L_n = F_{n-1} + F_{n+1}, \tag{6}$$

the relations

$$[F_{4h+v}/3] = \sum_{i=1}^{h/2} L_{8i+v-4} \quad (v = 0,\ 1,\ 2,\ 3;\ h \text{ even}) \tag{7}$$

$$[F_{4h+v}/3] = F_{v+1} + \sum_{i=1}^{(h-1)/2} L_{8i+v} \quad (v = 0,\ 1,\ 2;\ h \text{ odd}) \tag{7'}$$

$$[F_{4h+3}/3] = \sum_{i=1}^{(h+1)/2} L_{8i-5} \quad (h \text{ odd}) \tag{7''}$$

can be proven by induction on h. As an example, we consider the case h even and $v = 1$, and prove that

$$(F_{4h+1} - 1)/3 = \sum_{i=1}^{h/2} L_{8i-3}.$$

Setting $h = 2$, we obtain $(F_9 - 1)/3 = L_5$. Supposing the statement true for h, we have

$$\begin{aligned}\sum_{i=1}^{(h+2)/2} L_{8i-3} = \sum_{i=1}^{h/2+1} L_{8i-3} &= (F_{4h+1} - 1)/3 + L_{4h+5}\\ &= (F_{4h+1} - 1)/3 + F_{4h+4} + F_{4h+6}\\ &= (F_{4h+1} - 1)/3 + 18F_{4h} + 11F_{4h-1}\\ &= (34F_{4h+1} + 21F_{4h} - 1)/3\\ &= (F_{4h+9} - 1)/3 = (F_{4(h+2)+1} - 1)/3.\end{aligned}$$

From (7), (7′), (7″), and (6), it is seen that $[F_k/3]$ can be represented as a sum of $h + 1 = [k/4] + 1$ F-addends in the case $[k/4]$ odd and $k \equiv 3 \pmod 4$, and as a sum of $h = [k/4]$ F-addends otherwise.

Also solved (minus a closed form) by L. Kuipers and B. Poonen.

E Gads

H-394 *Proposed by Ambati Jaya Krishna, Baltimore, MD, and Gomathi S. Rao, Orangeburg, SC [Vol. 24(1), Feb. 1986]*

Find the value of the continued fraction $1 + \frac{2}{3+}\,\frac{4}{5+}\,\frac{6}{7+}\cdots$.

Solution by Paul S. Bruckman, Fair Oaks, CA

Define c_n, the n^{th} convergent of the indicated continued fraction, as follows:

(1) $$c_n \equiv u_n/v_n \equiv 1 + 2/3 + 4/5 + \cdots + 2n/(2n+1), \ n = 1, 2, \ldots;$$
$$c_0 \equiv 1 = 1/1.$$

After a moment's reflection, it is seen that u_n and v_n satisfy the common recurrence relation:

(2) $w_n = (2n+1)w_{n-1} + 2nw_{n-2}$, $n \geqslant 2$, where w_n denotes either u_n or v_n, and

(3) $u_0 = v_0 = 1;\ u_1 = 5,\ v_1 = 3.$

We now define the generating functions:

(4) $u(x) = \sum_{n=0}^{\infty} u_n \frac{x^n}{n!}$, $v(x) = \sum_{n=0}^{\infty} v_n \frac{x^n}{n!}$, $w(x)$ denoting either $u(x)$ or $v(x)$.

The initial conditions in (3) become:

(5) $u(0) = v(0) = 1;\ u'(0) = 5,\ v'(0) = 3.$

The recurrence in (2) translates to the following differential equation:

(6) $(2x - 1)w'' + (2x + 5)w' + 4w = 0.$

To solve (6), we find the following transformation useful:

(7) $g(x) = (2x - 1)w'(x) + 4w(x).$

Then, we find (6) is equivalent to the first-order homogeneous equation:

(8) $g' + g = 0,$

from which

(9) $g(x) = ae^{-x}$, for an unspecified constant a.

Substituting this last result into (7), after first making the transformation:

(10) $w(x) = h(x) \cdot (1 - 2x)^{-2},$

we find that $h'(x) = -a(1 - 2x)e^{-x}$, so

(11) $h(x) = -a(1 + 2x)e^{-x} + b$, where b is another unspecified constant.

Thus,

(12) $w(x) = (1 - 2x)^{-2}\{b - a(1 + 2x)e^{-x}\},$

where a and b are to be determined from (5), by appropriate differentiation in (12). Note that $w(0) = b - a = 1$. Also,

$$w'(x) = 4b(1-2x)^{-3} - 2ae^{-x}(1-2x)^{-3}(3+2x) + ae^{-x}(1+2x)(1-2x)^{-2},$$

so $w'(0) = 4b - 5a = 4 - a$. If $w(x) = u(x)$, then $a = -1$ and $b = 0$, while if $w(x) = v(x)$, then $a = 1$ and $b = 2$. Hence,

(13) $u(x) = (1 + 2x)(1 - 2x)^{-2}e^{-x}$, $v(x) = 2(1 - 2x)^{-2} - u(x).$

Next, we use (13) to obtain expansions for $u(x)$ and $v(x)$ and, therefore, explicit expressions for the u_n and v_n originally defined in (1). We start with

$$(1 + 2x)(1 - 2x)^{-2} = (1 + 2x)\sum_{n=0}^{\infty}(n + 1)2^n x^n$$

$$= \sum_{n=0}^{\infty}(n + 1)2^n x^n + \sum_{n=0}^{\infty} n2^n x^n = \sum_{n=0}^{\infty}(2n + 1)2^n x^n;$$

thus,

$$u(x) = \sum_{n=0}^{\infty}(2n + 1)2^n x^n \cdot \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \frac{(-1)^k}{k!}(2n - 2k + 1)2^{n-k}$$

$$= \sum_{n=0}^{\infty}(2n + 1)(2x)^n \sum_{k=0}^{n}\frac{(-\frac{1}{2})^k}{k!} - 2\sum_{n=1}^{\infty}(2x)^n \sum_{k=1}^{n}\frac{(-\frac{1}{2})^k}{(k - 1)!};$$

letting

(14) $$r_n = \sum_{k=0}^{n}\frac{(-\frac{1}{2})^k}{k!}, \quad n = 0, 1, 2, \ldots,$$

we obtain

$$u(x) = \sum_{n=0}^{\infty}(2n + 1)(2x)^n r_n + \sum_{n=1}^{\infty}(2x)^n\left(r_n - \frac{(-\frac{1}{2})^n}{n!}\right)$$

$$= 1 + \sum_{n=1}^{\infty}\left\{2(n + 1)r_n - \frac{(-\frac{1}{2})^n}{n!}\right\}(2x)^n,$$

or

(15) $$u(x) = \sum_{n=0}^{\infty}\left\{2^{n+1}(n + 1)!r_n - (-1)^n\right\}\frac{x^n}{n!}.$$

It follows from comparison of coefficients in (4) and (15) that

(16) $$u_n = 2^{n+1}(n + 1)!r_n - (-1)^n, \quad n = 0, 1, 2, \ldots .$$

Likewise, since $v(x) = 2(1 - 2x)^{-2} - u(x)$, we find

$$v(x) = 2\sum_{n=0}^{\infty}(n + 1)2^n x^n - \sum_{n=0}^{\infty} u_n \frac{x^n}{n!} = \sum_{n=0}^{\infty}(n + 1)!2^{n+1}\frac{x^n}{n!} - \sum_{n=0}^{\infty} u_n \frac{x^n}{n!},$$

so

(17) $$v_n = 2^{n+1}(n + 1)! - u_n,$$

or

(18) $$v_n = 2^{n+1}(n + 1)!(1 - r_n) + (-1)^n, \quad n = 0, 1, 2, \ldots .$$

We note that

(19) $$\lim_{n\to\infty} r_n = e^{-\frac{1}{2}}.$$

Therefore,

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty}(u_n/v_n) = \lim_{n\to\infty}\left\{\frac{2^{n+1}(n + 1)!r_n - (-1)^n}{2^{n+1}(n + 1)!(1 - r_n) + (-1)^n}\right\} = \lim_{n\to\infty}\left(\frac{r_n}{1 - r_n}\right)$$

$$= e^{-\frac{1}{2}}/(1 - e^{-\frac{1}{2}}),$$

or

(20) $$\lim_{n\to\infty} c_n = (e^{\frac{1}{2}} - 1)^{-1} \doteq 1.541494083.$$

Also solved by W. Janous, A. Krishna & G. Rao, L. Kuipers & P. Shieu, J.-S. Lee & J.-Z. Lee, F. Steutel, and the proposer.

Easy Induction

H-395 *Proposed by Heinz-Jürgen Seiffert, Berlin Germany*
[Vol. 24(1), Feb. 1986]

Show that for all positive integers m and k,

$$\sum_{n=0}^{m-1} \frac{F_{2k(2n+1)}}{L_{2n+1}} = \sum_{j=0}^{k-1} \frac{F_{2m(2j+1)}}{L_{2j+1}}$$

Solution by J.-Z. Lee & J.-S. Lee, Soochow University, Taipei, Taiwan, R.O.C.

Define

$$S_1(m, k) = \sum_{n=0}^{m-1} (F_{2k(2n+1)}/L_{2n+1}),$$

$$S_2(m, k) = \sum_{j=0}^{k-1} (F_{2m(2j+1)}/L_{2j+1}).$$

From the definitions of F_n and L_n, we have

Lemma 1: $F_{(m+2k)(2n+1)} - F_{m(2n+1)} = F_{(m+k)(2n+1)}L_{k(2n+1)}$,

Lemma 2: $\sum_{n=0}^{m-1} F_{(2k-1)(2n+1)} = F_{2m(2k-1)}/L_{2k-1}$.

We will prove, using the induction hypothesis, that

$$S_1(m, k) = S_2(m, k) \qquad (*)$$

for all positive integers m and k.

For $k = 1$, we obtain

$$S_1(m, 1) = \sum_{n=0}^{m-1} (F_{2(2n+1)}/L_{2n+1}) = \sum_{n=0}^{m-1} F_{2n+1} = F_{2m} = S_2(m, 1),$$

so $(*)$ is true for $k = 1$. Suppose that $(*)$ is true for all positive integers less than k, then

$$\begin{aligned} S_1(m, k) &= \sum_{n=0}^{m-1} (F_{2k(2n+1)}/L_{2n+1}) \\ &= \sum_{n=0}^{m-1} ((F_{2(k-1)(2n+1)} + F_{(2k-1)(2n+1)}L_{2n+1})/L_{2n+1}), \text{ by Lemma 1,} \\ &= \sum_{n=0}^{m-1} (F_{2(k-1)(2n+1)}/L_{2n+1}) + \sum_{n=0}^{m-1} F_{(2k-1)(2n+1)} \\ &= \sum_{j=0}^{k-2} (F_{2m(2j+1)}/L_{2j+1}) + F_{2m(2k-1)}/L_{2k-1}, \\ &\qquad \text{by the induction hypothesis and Lemma 2,} \\ &= \sum_{j=0}^{k-1} (F_{2m(2j+1)}/L_{2j+1}) = S_2(m, k); \end{aligned}$$

therefore, (*) is true for all positive integers k.

Also solved by P. Bruckman, L. A. G. Dresel, C. Georghiou, W. Janous, L. Kuipers, and the proposer.

Another Easy One

H-396 *Proposed by M. Wachtel, Zürich, Switzerland [Vol. 24(1), Feb. 1986]*

Establish the identity:

$$\sum_{i=1}^{\infty} \frac{F_{i+n}}{a^i} + \sum_{i=1}^{\infty} \frac{F_{i+n+1}}{a^i} = \sum_{i=1}^{\infty} \frac{F_{i+n+2}}{a^i}$$

$$a = 2, 3, 4, \ldots, n = 0, 1, 2, 3, \ldots .$$

Solution by Paul S. Bruckman, Fair Oaks, CA

The series defined as follows,

$$f(x, m) \equiv \sum_{i=1}^{\infty} F_{i+m} x^i, \; m \in Z, \tag{1}$$

is absolutely convergent, with radius of convergence $\theta \equiv \frac{1}{2}(\sqrt{5} - 1) \doteq .618$. In fact, the sum of the series is readily found to be equal to

$$f(x, m) = \frac{xF_{m+1} + x^2 F_m}{1 - x - x^2}, \quad |x| < \theta. \tag{2}$$

Since $a^{-1} < \theta$ for $a = 2, 3, 4, \ldots$, each of the series indicated in the statement of the problem is absolutely convergent. Hence,

$$\sum_{i=1}^{\infty} F_{i+n} a^{-i} + \sum_{i=1}^{\infty} F_{i+n+1} a^{-i} = \sum_{i=1}^{\infty} (F_{i+n} + F_{i+n+1}) a^{-i} = \sum_{i=1}^{\infty} F_{i+n+2} a^{-i}.$$

This may also be demonstrated from (2), setting $x = a^{-1}$:

$$f(a^{-1}, n) + f(a^{-1}, n + 1) = \frac{aF_{n+1} + F_n}{a^2 - a - 1} + \frac{aF_{n+2} + F_{n+1}}{a^2 - a - 1}$$

$$= \frac{aF_{n+3} + F_{n+2}}{a^2 - a - 1} = f(a^{-1}, n + 2).$$

Also solved by L. A. G. Dresel, P. Filipponi, C. Georghiou, W. Janous, L. Kuipers, J.-Z. Lee & J.-S. Lee, R. Whitney, and the proposer.

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