FRIENDLY-PAIRS OF MULTIPLICATIVE FUNCTIONS

N. BALASUBRAHMANYAN
Joint Cipher Bureau, D1 Block, Sena Bhavan, New Delhi 110011, India
R. SIVARAMAKRISHNAN
University of Calicut, Calicut 673 635, Kerala, India

(Submitted October 1985)

1. INTRODUCTION

An arithmetic function f is said to be multiplicative if

$$f(m)f(n) = f(mn) \text{ whenever } (m, n) = 1. \tag{1.1}$$

It is a consequence of (1.1) that f is known if $f(p^r)$ is known for every prime p and $r \geqslant 1$.

Definition: A pair $\{f, g\}$ of multiplicative functions is called a "friendly-pair" of the type α ($\alpha \geqslant 2$) if, for $n \geqslant 1$,

$$f(n^\alpha) = g(n), \quad g(n^\alpha) = f(n) \tag{1.2}$$

and

$$f(n)g(n) = 1. \tag{1.3}$$

Question: Do friendly-pairs of multiplicative functions exist?

We answer this question in the affirmative.

2. A FRIENDLY-PAIR

We exhibit a friendly-pair of multiplicative functions by actual construction. As f, g are multiplicative, it is enough if we work with prime-powers.

Let p be a prime and $r \geqslant 1$.

We define f and g by the expressions:

$$f(p^r) = \exp\left(\frac{2\pi ik}{\alpha + 1}\right) \text{ if } r \equiv k \pmod{(\alpha + 1)} \tag{2.1}$$

$$g(p^r) = \exp\left(\frac{-2\pi ik}{\alpha + 1}\right) \text{ if } r \equiv k \pmod{(\alpha + 1)} \tag{2.2}$$

We immediately deduce that

$$f(p^{r\alpha}) = \exp\left(\frac{2\pi ik\alpha}{\alpha + 1}\right) = \exp\left(\frac{-2\pi ik}{\alpha + 1}\right) = g(p^r).$$

Similarly, we obtain

$$g(p^{r\alpha}) = f(p^r).$$

Therefore, we get

$$f(n^\alpha) = g(n) \quad \text{and} \quad g(n^\alpha) = f(n).$$

Also, $f(p^{\alpha+1}) = g(p^{\alpha+1}) = 1$. Thus, from (2.1) and (2.2), we obtain

$$f(p^r)g(p^r) = 1, \ r \geqslant 1.$$

Or, $f(n)$ and $g(n)$ are such that $f(n)g(n) = 1$.

Example: For $\alpha = 2$, we note that f, g would form a friendly-pair satisfying

$$f(n^2) = g(n), \ g(n^2) = f(n), \text{ and } f(n)g(n) = 1, \ n \geqslant 1.$$

In this case, f and g are given by:

$$f(p^r) = \begin{cases} \exp(2\pi i/3) & \text{if } r \equiv 1 \pmod 3 \\ \exp(4\pi i/3) & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod{\ } \end{cases} \tag{2.3}$$

$$g(p^r) = \begin{cases} \exp(-2\pi i/3) & \text{if } r \equiv 1 \pmod 3 \\ \exp(-4\pi i/3) & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.4}$$

Before concluding, we remark that there exist pairs $\{f, g\}$ which satisfy (1.2) but not (1.3). This point is elucidated for the case $\alpha = 2$.

Let $\mu(n)$ be the Möbius function. We define $f(n)$ and $g(n)$ as follows:

$$f(n) = \sum_{n = dt^3} \mu(d), \tag{2.5}$$

where the summation is over the divisors d of n for which the complementary divisor n/d is a perfect cube.

$$g(n) = \sum_{n = d^2t^3} \mu(d),$$

where the summation is over the square divisors d^2 of n for which the complementary divisor n/d^2 is a perfect cube.

We observe that f and g are multiplicative. Further,

$$f(p^r) = \begin{cases} -1 & \text{if } r \equiv 1 \pmod 3 \\ 0 & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.7}$$

$$g(p^r) = \begin{cases} 0 & \text{if } r \equiv 1 \pmod 3 \\ -1 & \text{if } r \equiv 2 \pmod 3 \\ 1 & \text{if } r \equiv 0 \pmod 3 \end{cases} \tag{2.8}$$

It is easy to check that $f(n^2) = g(n)$ and $g(n^2) = f(n)$ for $n \geqslant 1$. However,

$$f(n)g(n) = \begin{cases} 1 & \text{if } n \text{ is a perfect cube,} \\ 0 & \text{otherwise.} \end{cases}$$

This pair $\{f, g\}$ is not a friendly-pair.

◆◇◆◇◆