GENERALIZED STIRLING NUMBER PAIRS ASSOCIATED WITH INVERSE RELATIONS

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1. INTRODUCTION

Stirling numbers and some of their generalizations have been investigated Intensively during the past several decades. Useful references for various results may be found in [1], [2, ch. 5], [3], [6], [7], etc.

The main object of this note is to show that the concept of a generalized Stirling number pair can be characterized by a pair of inverse relations. Our basic idea is suggested by the well-known inverse relations as stated explicitly in Riordan's classic book [7], namely

$$a_n = \sum_{k=0}^{n} S_1(n, k)b_k, \qquad b_n = \sum_{k=0}^{n} S_2(n, k)a_k,$$

where $S_1(n, k)$ and $S_2(n, k)$ are Stirling numbers of the first and second kind, respectively. Recall that $S_1(n, k)$ and $S_2(n, k)$ may be defined by the exponential generating functions

$$(\log(1 + t))^k/k! \quad \text{and} \quad (e^t - 1)^k/k!,$$

respectively, where

$$f(t) = \log(1 + t) \quad \text{and} \quad g(t) = e^t - 1$$

are just reciprocal functions of each other, namely $f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$. What we wish to elaborate is a comprehensive generalization of the known relations mentioned above.

2. A BASIC DEFINITION AND A THEOREM

Denote by $\Gamma \equiv (\Gamma, +, \cdot)$ the commutative ring of formal power series with real or complex coefficients, in which the ordinary addition and Cauchy multiplication are defined. Substitution of formal power series is defined as usual (cf. Comtet [2]).

Two elements f and g of Γ are said to be reciprocal (inverse) of each other if and only if $f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$.

Definition: Let f and g belong to Γ, and let

$$\frac{1}{k!}(f(t))^k = \sum_{n\geqslant 0} A_1(n, k)\frac{t^n}{n!}, \tag{2.1}$$

$$\frac{1}{k!}(g(t))^k = \sum_{n\geqslant 0} A_2(n, k)\frac{t^n}{n!}. \tag{2.2}$$

Then $A_1(n, k)$ and $A_2(n, k)$ are called a generalized Stirling number pair, or a GSN pair if and only if f and g are reciprocal of each other.

From (2.1) and (2.2), one may see that every GSN pair has the property

$$A_1(n, k) = A_2(n, k) = 0 \text{ for } n < k.$$

Moreover, one may define

$$A_1(0, 0) = A_2(0, 0) = 1.$$

Let us now state and prove the following:

Theorem: Numbers $A_1(n, k)$ and $A_2(n, k)$ defined by (2.1) and (2.2) just form a GSN pair when and only when there hold the inverse relations

$$a_n = \sum_{k=0}^{n} A_1(n, k)b_k, \qquad b_n = \sum_{k=0}^{n} A_2(n, k)a_k, \tag{2.3}$$

where $n = 0, 1, 2, \ldots,$ and either $\{a_k\}$ or $\{b_k\}$ is given arbitrarily.

Proof: We have to show that $(2.3) \Longleftrightarrow f(g(t)) = g(f(t)) = t$ with $f(0) = g(0) = 0$. As may easily be verified, the necessary and sufficient condition for (2.3) to hold is that the orthogonality relations

$$\sum_{n \geq 0} A_1(m, n)A_2(n, k) = \sum_{n \geq 0} A_2(m, n)A_1(n, k) = \delta_{mk}, \tag{2.4}$$

hold, where δ_{mk} is the Kronecker symbol. Clearly, both summations contained in (2.4) consist of only a finite number of terms inasmuch as

$$A_1(m, n) = A_2(m, n) = 0 \text{ for } n > m.$$

Let us prove $\Longrightarrow$. Since (2.4) is now valid, we may substitute (2.1) into (2.2), and by the rule of function composition we obtain

$$\begin{aligned}\frac{1}{k!}(g(f(t)))^k &= \sum_{n \geq 0} A_2(n, k) \sum_{m \geq 0} A_1(m, n)\frac{t^m}{m!} \\ &= \sum_{m \geq 0} \frac{t^m}{m!}\left(\sum_{n \geq 0} A_1(m, n)A_2(n, k)\right) = \sum_{m \geq 0} \frac{t^m}{m!}\,\delta_{mk} = \frac{t^k}{k!}.\end{aligned}$$

Thus, it follows that $g(f(t)) = t$. Similarly, we have $f(g(t)) = t$. This proves $\Longrightarrow$.

To prove $\Longleftarrow$, suppose that $f(g(t)) = g(f(t)) = t$, $f(0) = g(0) = 0$. Substituting (2.2) into (2.1), we obtain

$$\frac{1}{k!}\,t^k = \frac{1}{k!}(f(g(t)))^k = \sum_{m \geq 0} \frac{t^m}{m!}\left(\sum_{n \geq 0} A_2(m, n)A_1(n, k)\right).$$

Comparing the coefficients of t on both sides, we get

$$\sum_{n \geq 0} A_2(m, n)A_1(n, k) = \delta_{mk}.$$

In a similar manner, the first equation contained in (2.4) can be deduced. Recalling that (2.4) is precisely equivalent to (2.3), the inverse implication $\Longleftarrow$ is also verified; hence, the theorem.

Evidently, the theorem just proved may be restated as follows:

Equivalence Proposition: The following three assertions are equivalent to each other.

(i) $\{A_1(n, k), A_2(n, k)\}$ is a GSN pair.

(ii) Inverse relations (2.3) hold.

(iii) $\{f, g\}$ is a pair of reciprocal functions of Γ.

3. EXAMPLES AND REMARKS

Examples: Some special GSN pairs may be displayed as shown below.

$f(t)$	$g(t)$	$A_1(n, k)$	$A_2(n, k)$
$\log(1 + t)$	$e^t - 1$	$S_1(n, k)$	$S_2(n, k)$
$\tan t$	$\text{arc tan } t$	$T_1(n, k)$	$T_2(n, k)$
$\sin t$	$\text{arc sin } t$	$\bar{s}_1(n, k)$	$\bar{s}_2(n, k)$
$\sinh t$	$\text{arc sinh } t$	$\sigma_1(n, k)$	$\sigma_2(n, k)$
$\tanh t$	$\text{arc tanh } t$	$\tau_1(n, k)$	$\tau_2(n, k)$
$t/(t - 1)$	$t/(t - 1)$	$(-1)^{n-k}L(n, k)$	$(-1)^{n-k}L(n, k)$

Note that $L(n, k)$ is known as Lah's number, which has the expression

$$L(n, k) = (-1)^n \frac{n!}{k!}\binom{n - 1}{k - 1}.$$

In what follows, we will give a few brief remarks that follow easily from the ordinary theory about exponential generating functions.

Remark 1: For a pair of reciprocal elements $f, g \in \Gamma$, write:

$$f(t) = \sum_1^\infty \alpha_k t^k/k!, \qquad g(t) = \sum_1^\infty \beta_k t^k/k! \tag{3.1}$$

Making use of the definition of Bell polynomials (cf. Riordan [7]),

$$Y_n(gf_1, \ldots, gf_n) = \sum_{(J)} \frac{n! g_k}{j_1! \ldots j_n!}\left(\frac{f_1}{1!}\right)^{j_1} \cdots \left(\frac{f_n}{n!}\right)^{j_n},$$

where (J) indicates the summation condition $j_1 + \cdots + j_n = k$, $1j_1 + 2j_2 + \cdots + nj_n = n$, $k = 1, 2, \ldots, n$, one may obtain

$$A_1(n, k) = Y_n(f\alpha_1, \ldots, f\alpha_n), \qquad A_2(n, k) = Y_n(f\beta_1, \ldots, f\beta_n),$$

where $f_i = \delta_{ki}$ $(i = 1, \ldots, n)$ and δ_{ki} is the Kronecker symbol. Consequently, certain combinatorial probabilistic interpretation may be given of $A_i(n, k)$ $(i = 1, 2)$. Moreover, for any given $\{\alpha_k\}$, the sequence $\{\beta_k\}$ can be determined by the system of linear equations

$$Y_n(\beta\alpha_1, \ldots, \beta\alpha_n) = \delta_{n1} \qquad (n = 1, 2, \ldots). \tag{3.2}$$

Remark 2: It is easy to write down double generating functions for $A_i(n, k)$, viz.,

$$\Phi(t, u) := \sum_{n, k \geqslant 0} A_1(n, k)\frac{t^n u^k}{n!} = \exp[uf(t)],$$

$$\Psi(t,\ u) := \sum_{n,\,k \geqslant 0} A_2(n,\ k)\frac{t^n u^k}{n!} = \exp[ug(t)].$$

Moreover, for each $A_i(n,\ k)$ $(i = 1,\ 2)$, we have the convolution formula

$$\binom{k_1 + k_2}{k_1} A_i(n,\ k_1 + k_2) = \sum_{j=0}^{n} \binom{n}{j} A_i(j,\ k_1) A_i(n - j,\ k_2), \tag{3.3}$$

and, consequently, there is a vertical recurrence relation for $A_i(n,\ k)$, viz.,

$$kA_i(n,\ k) = \sum_{j=0}^{n-1} \binom{n}{j} A_i(j,\ k - 1) A_i(n - j,\ 1), \tag{3.4}$$

where $A_1(j,\ 1) = \alpha_j$ and $A_2(j,\ 1) = \beta_j$. A similar recurrence relation takes the form

$$A_i(n + 1,\ k) = \sum_{j=0}^{n} \binom{n}{j} A_i(j,\ k - 1) A_i(n - j + 1,\ 1). \tag{3.5}$$

However, we have not yet found any useful horizontal recurrence relations for $A_i(n,\ k)$ $(i = 1,\ 2)$. Also unsolved are the following:

Problems: How to determine some general asymptotic expansions for $A_i(n,\ k)$ as $k \to \infty$ with $k = o(n)$ or $k = O(n)$? Is it true that the asymptotic normality of $A_1(n,\ k)$ implies that of $A_2(n,\ k)$? Is it possible to extend the concept of a GSN pair to a case involving multiparameters?

4. A CONTINUOUS ANALOGUE

We are now going to extend, in a similar manner, the reciprocity of the relations (2.3) to the case of reciprocal integral transforms so that a kind of GSN pair containing continuous parameters can be introduced.

Let $\phi(x)$ and $\psi(x)$ be real-valued reciprocal functions decreasing on $[0,\ 1]$ with $\phi(0) = \psi(0) = 1$ and $\phi(1) = \psi(1) = 0$, such that

$$\phi(\psi(x)) = \psi(\phi(x)) = x \qquad (0 \leqslant x \leqslant 1).$$

Moreover, $\phi(x)$ and $\psi(x)$ are assumed to be infinitely differentiable in $(0,\ 1)$. Introduce the substitution $x = e^{-t}$, so that we may write

$$e^{-u} = \phi(e^{-t}), \quad e^{-t} = \psi(e^{-u}), \quad t, u \in [0,\ \infty). \tag{4.1}$$

For given measurable functions $f(s) \in L(0,\ \infty)$, consider the integral equation

$$F(u) := \int_0^\infty f(s) e^{-us}\, ds = \int_0^\infty g(s)(\psi(e^{-u})^s)\ ds, \tag{4.2}$$

where $g(s)$ is to be determined. Evidently, (4.2) is equivalent to the following:

$$G(t) := \int_0^\infty f(s)(\phi(e^{-t}))^s\, ds = \int_0^\infty g(s) e^{-ts}\, ds. \tag{4.3}$$

Denote $G(t) = F(u) = F(-\log \phi(e^{-t}))$. Suppose that $G(t)$ satisfies the Widder condition D (cf. [8], ch. 7, §6, §17):

(i) $G(t)$ is infinitely differentiable in $(0,\ \infty)$ with $G(\infty) = 0$.

(ii) For every integer $m \geqslant 1$, $L_{m,x}[G] \equiv L_{m,x}[G(\cdot)]$ is Lebesgue integrable on $(0, \infty)$, where $L_{m,x}[G]$ is the Post-Widder operator defined by

$$L_{m,x}[G] := \frac{(-1)^m}{m!}\left(\frac{m}{x}\right)^{m+1}\left(\frac{d}{dt}\right)^m G(t)\Big|_{t=(m/x)}. \tag{4.4}$$

(iii) The sequence $\{L_{m,x}[G]\}$ converges in mean of index unity, namely

$$\lim_{m,n\to\infty}\int_0^\infty |L_{m,x}[G] - L_{n,x}[G]|dx = 0.$$

Then by the representation theorem of Widder (cf. [8], Theorem 17, p. 318) one may assert the existence of $g(s) \in L(0,\infty)$ such that (4.3) holds. Consequently, the well-known inversion theorem of Post-Widder (*loc. cit.*) is applicable to both (4.3) and (4.2), yielding

$$g(x) = \lim_{m\to\infty}\int_0^\infty f(s)L_{m,x}[(\phi(e^{-(\cdot)}))^s]ds, \tag{4.5}$$

$$f(x) = \lim_{m\to\infty}\int_0^\infty g(s)L_{m,x}[(\psi(e^{-(\cdot)}))^s]ds, \tag{4.6}$$

whenever $x > 0$ belongs to the Lebesgue sets of g and f, respectively.

In fact, the reciprocity (4.5) $\Longleftrightarrow$ (4.6) so obtained is just a generalization of the inverse relations for self-reciprocal integral transforms (in the case $\phi \equiv \psi$) discussed previously (cf. [4], Theorem 8).

Notice that $A_i(n, k)$ ($i = 1, 2$) may be expressed by using formal derivatives:

$$A_1(n, k) = \frac{1}{k!}\left(\frac{d}{dt}\right)^n (f(t))^k\Big|_{t=0}, \qquad A_2(n, k) = \frac{1}{k!}\left(\frac{d}{dt}\right)^n (g(t))^k\Big|_{t=0}.$$

Thus, recalling (4.4) and comparing (4.5) and (4.6) with (2.3), it seems to be reasonable to consider the following two sequences of numbers:

$$A_1^*(x, y; m) = L_{m,x}[(\phi(e^{-(\cdot)}))^y],$$
$$A_2^*(x, y; m) = L_{m,x}[(\psi(e^{-(\cdot)}))^y] \qquad (m = 1, 2, \ldots),$$

as a kind of GSN pair involving continuous parameters $x, y \in (0, \infty)$.

In conclusion, all we have shown is that the continuous analogue of the concept for a GSN pair is naturally connected to a general class of reciprocal integral transforms. Surely, special reciprocal functions $\phi(x)$ and $\psi(x)$ ($0 \leqslant x \leqslant 1$) may be found—as many as one likes. For instance, if one takes

$$\phi_1(x) = 1 - x, \quad \phi_2(x) = \cos\frac{\pi x}{2}, \quad \phi_3(x) = \log(e - (e - 1)x),$$

their corresponding inverse functions are given by

$$\psi_1(x) = 1 - x, \quad \psi_2(x) = \frac{2}{\pi}\operatorname{arc\,cos} x, \quad \psi_3(x) = (e - e^x)/(e - 1),$$

respectively. Monotone and boundary conditions

$$\phi_i(0) = \psi_i(0) = 1 \text{ and } \phi_i(1) = \psi_i(1) = 0 \qquad (i = 1, 2, 3)$$

are obviously satisfied.

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