ON THE LARGEST ODD COMPONENT OF A UNITARY PERFECT NUMBER*

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1. INTRODUCTION

A divisor d of an integer n is a *unitary divisor* if gcd $(d, n/d) = 1$. If d is a unitary divisor of n we write $d \| n$, a natural extension of the customary notation for the case in which d is a prime power. Let $\sigma^*(n)$ denote the sum of the unitary divisors of n:

$$\sigma^*(n) = \sum_{d \| n} d.$$

Then σ^* is a multiplicative function and $\sigma^*(p^e) = 1 + p^e$ for p prime and $e > 0$.

We say that an integer N is *unitary perfect* if $\sigma^*(N) = 2N$. In 1966, Subbaro and Warren [2] found the first four unitary perfect numbers:

$$6 = 2 \cdot 3;\ 60 = 2^2 3 \cdot 5;\ 90 = 2 \cdot 3^2 5;\ 87{,}360 = 2^6 3 \cdot 5 \cdot 7 \cdot 13.$$

In 1969, I announced [3] the discovery of another such number,

$$146{,}361{,}936{,}186{,}458{,}562{,}560{,}000$$
$$= 2^{18} 3 \cdot 5^4 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

which I later proved [4] to be the fifth unitary perfect number. No other unitary perfect numbers are known.

Throughout what follows, let $N = 2^a m$ (with m odd) be unitary perfect and suppose that K is the largest odd component (i.e., prime power unitary divisor) of N. In this paper we outline a proof that, except for the five known unitary perfect numbers, $K > 2^{15}$.

2. TECHNIQUES

In light of the fact that $\sigma^*(p^e) = 1 + p^e$ for p prime, the problem of finding a unitary perfect number is equivalent to that of expressing 2 as a product of fractions, with each numerator being 1 more than its denominator, and with the denominators being powers of distinct primes. If such an expression for 2 exists, then the denominator of the unreduced product of fractions is unitary perfect. The main tool is the epitome of simplicity: we must eventually divide out any odd prime that appears in either a numerator or a denominator.

If p is an odd prime, then $\sigma^*(p^e) = 1 + p^e$ is even. Thus, if some of the odd components of a unitary perfect number N are known or assumed, there is an implied lower bound for a, where $2^a \| N$, since all but one of the 2's in the numerator of $\sigma^*(N)/N$ must divide out. Another lower bound, useful in many cases, is Subbarao's result [1] that $a > 10$ except for the first four unitary perfect numbers.

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A simple program was run on a microcomputer to find, for each odd prime $p < 2^{15}$, the smallest A for which $2^A \equiv \pm 1 \pmod{p}$. If $2^A \equiv 1 \pmod{p}$, then p never divides $1 + 2^a$. If $2^A \equiv -1 \pmod{p}$, then p divides $1 + 2^a$ if and only if a is an odd integer times A, and we refer to A as the *entry point* of p.

If an odd prime p has entry point A and $p^2 | (1 + 2^A)$, it is easy to see that $2^{p-1} \equiv 1 \pmod{p^2}$. There are only two primes less than $3 \cdot 10^9$ for which this this phenomenon occurs, and they are 1093 and 3511. Then $1 + 2^A$ would have a component larger than 10^6. Thus, for the primes $p < 2^{15}$ under consideration here, either p never divides $1 + 2^a$ or $p \| (1 + 2^A)$ or $1 + 2^a$ has a component larger than 2^{15}.

The odd primes less than 2^{15} having entry points were ordered by entry point. Then it was a fairly easy procedure to consider algebraic factors and conclude that $1 + 2^a$ has all components less than 2^{15} for only $a < 11$ and the a shown in Table 1.

Table 1

2^a	$1 + 2^a$	2^{24}	97*257*673
2^{11}	3*683	2^{25}	3*11*251*4051
2^{12}	17*241	2^{26}	5*53*157*1613
2^{13}	3*2731	2^{30}	5^2*13*41*61*1321
2^{14}	5*29*113	2^{33}	3^2*67*683*20857
2^{15}	3^2*11*331	2^{34}	5*137*953*26317
2^{18}	5*13*37*109	2^{42}	5*13*29*113*1429*14449
2^{21}	3^2*43*5419	2^{46}	5*277*1013*1657*30269
2^{22}	5*397*2113	2^{78}	5*13^2*53*157*313*1249*1613*3121*21841

In many of the proofs, cases are eliminated because under the stated conditions $\sigma^*(N)/N$ would be less than 2. A number n for which $\sigma^*(n) < 2n$ is called *unitary deficient* (abbreviated "u-def"). Finally, we will write $a = A \cdot$ odd to indicate that a is an odd integer times A.

3. PRELIMINARY CASES

If $K = 3$, we have $3 | \sigma^*(2^a)$, so a is odd. But N is u-def if $a \geq 3$, so $a = 1$; hence, $N = 2 \cdot 3 = 6$, the first unitary perfect number.

If $K = 5$, we immediately have $3 \| N$ and $a = 2 \cdot$ odd. But N is u-def if $a \geq 6$, so $a = 2$; therefore, $N = 2^2 3 \cdot 5 = 60$, the second unitary perfect number.

Note that $K = 7$ is impossible, because 7 never divides $1 + 2^a$. In general, the largest component cannot be the first power of a prime that has no entry point.

If $K = 3^2 = 9$, then $5 \| N$, and $\sigma^*(5)$ uses one of the two 3's. To use the other 3, we must have $3 | \sigma^*(2^a)$, so a is odd. Now, $7 \nmid N$ or else $7 | \sigma^*(2^a)$, which is impossible. Then N is u-def if $a \geq 3$, so $a = 1$; hence, $N = 2 \cdot 3^2 5 = 90$, the third unitary perfect number.

If $K = 11$, then $11 | \sigma^*(2^a)$, so $a = 5$ odd; hence, $3 | \sigma^*(2^a)$. But $3 | \sigma^*(11)$ as well, so $3^2 \| N$. Then $5 | \sigma^*(3^2)$, so $5 \| N$, and since $3 | \sigma^*(5)$ we have $3^3 | N$, contradicting the maximality of K.

If $K = 13$, we have $13 | \sigma^*(2^a)$, so $a = 6$ odd; hence, $5 | \sigma^*(2^a)$. Then $5 \| N$, so $3 \| N$ because $3^2 \| N$ would imply $5^2 | N$, a contradiction. Because $13 \| N$, we have $7 \| N$, but we cannot have $11 \| N$ or else $3^2 | N$. But N is u-def if $a \geqslant 18$, so $a = 6$, from which it follows that $N = 2^6 3 \cdot 5 \cdot 7 \cdot 13 = 87{,}360$, the fourth unitary perfect number.

We have now accounted for the first four unitary perfect numbers. In light of Subbarao's results [1], we may assume that $a > 10$ from now on.

Now suppose $a = 78$. Because $313 \cdot 1249 | \sigma^*(2^{78})$ and the squares of these primes exceed 2^{15}, we have $313 \cdot 1249 \| N$. But $157^2 | \sigma^*(2^{78} 313)$, so $157^2 \| N$. However, $5^7 | \sigma^*(2^{78} 157^2 1249)$, so $5^7 | N$. But $5^7 > 2^{15}$, so $a = 78$ is impossible.

At this stage, a table was constructed to list all odd prime powers which might be components in the remaining cases. For the sake of brevity, the table and most of the remaining proofs are omitted here. However, the table may be obtained from the author. The table was constructed to include: (1) the odd primes that appear in Table 1 (except for $a = 78$); (2) all odd primes dividing $\sigma^*(q)$, where q is any other number also in Table 2 below; and (3) all allowable powers of primes also in Table 2. A "possible sources" column listed all components of unreduced denominators in $\sigma^*(N)/N$ for which a particular prime might appear in a numerator; multiple appearances were also indicated.

Insufficient entries in the "possible sources" column allow us to eliminate some possible components. For example, there are only two possible sources for 23, so 23^3 cannot occur. We eliminate: 23^3; 31^2; 31^3; 67^2 and hence 449; 71^2 and hence 2521; 73^2; in succession, 79^2, 3121, and 223; successively, 101^2, 5101, and 2551; successively, 131^2, 8581, 613, and 307; successively, 139^2, 9661, 4831, 151^2, 877, and 439; successively, 149^2, 653, 109^2, 457, 229, and 23^2; and successively, 181^2, 16381, and 8191.

4. REMAINING CASES

We have $11 \leqslant a \leqslant 46$, so there can be no more than 47 odd components. The smallest odd component must be smaller than 17 because a $\sigma^*(N)/N$ ratio of 1.926... occurs if N is the product of 2^{11} and the following 47 prime powers:

17, 19, 23, 25, 27, 29, 31, 37, 41, 43, 47, 49, 53, 61, 67, 73,
79, 83, 97, 101, 109, 113, 121, 131, 137, 139, 149, 151, 157,
169, 181, 191, 193, 199, 211, 241, 251, 257, 269, 271, 277,
281, 313, 331, 337, 397, 421

If $83^2 \| N$, then $331 \cdot 829 \| N$. If 829 is a component, then 1657 is also, and hence $a = 46$. Now, 331 is a component only if $a = 15$ or $661 \| N$, and since $a = 46$, $661 \| N$. But then $1321 \| N$, so $a = 30$, a contradiction. Therefore, 83^2 cannot be a component.

Suppose $a = 46$. Then $277 \cdot 1013 \cdot 1657 \cdot 30269 \| N$, so $139 \cdot 829 \cdot 1009 \| N$, hence $83 \cdot 101 \| N$. Therefore, $3^4 5^5 7^2 13^2 | N$, so $11 \| N$, because there must be a component smaller than 17, and $\sigma^*(11)$ contributes another 3 to the numerator of $\sigma^*(N)/N$. Now, either $5^5 \| N$ or $5^6 \| N$. If $5^5 \| N$, then $521 \| N$ and we have, successively, 29^2, 421, 211, and 53 as components; but then $3^{10} | N$, which is impossible. Thus, $5^6 \| N$, so $601 \| N$, hence $43 | N$. But $43 \| N$ or else there are too many 5's. Now, $7^3 \| N$ would force $43^2 | N$, and $7^4 \| N$ would force $1201 \| N$, hence $601^2 \| N$, so $7^5 \| N$; however, then $11^2 | N$, a contradiction. Therefore, we may eliminate $a = 46$. As a result, we may eliminate 277, 1657, 829, and 30269 as components, then 139 and 1009, and then 101.

For the sake of brevity, the other cases, except $a = 24$, are summarized in Table 2.

Table 2

CASE	ELIMINATED	CASE	ELIMINATED
$2^{42}*17^3$	$2^{42}*17^3$	$2^{12}*11^4$	$2^{12}*11^4$
$2^{42}*7$	$2^{42}*7$	$2^{12}*11^3$	$2^{12}*11^3$
2^{42}	2^{42}; 113^2; 1277; 71	2^{12}	2^{12}
2^{26}	2^{26}	$2^{21}*43^2$	$2^{21}*43^2$
53^2	53^2; 281; 47^2	$2^{21}*5$	$2^{21}*5$
2^{34}	2^{34}; 26317; 13159; 47	2^{21}	2^{21}
2^{33}	2^{33}; 67	61^2	61^2; 1861
41^2	41^2	19^3	19^3
$2^{30}*61$	$2^{30}*61$	$2^{22}*19^2$	$2^{22}*19^2$
2^{30}	2^{30}	2^{22}	2^{22}
2^{25}	2^{25}	$2^{18}*37^2$	$2^{18}*37^2$
2^{11}	2^{11}	$2^{18}*19^2$	$2^{18}*19^2$
2^{13}	2^{13}	$2^{18}*5^3$	$2^{18}*5^3$
2^{15}	2^{15}; 441; 83	$2^{18}*5^5$	$2^{18}*5^5$
$2^{14}*29^2$	$2^{14}*29^2$	$2^{18}*5^6$	$2^{18}*5^6$
2^{14}	2^{14}; 113	2^{18}	2^{18} unless $N = W$; 109

The ordering of cases presented in Table 2 works fairly efficiently. The reader should rest assured that sudden departures from an orderly flow are deliberate and needed. The case $\alpha = 24$ is especially difficult, and so is presented here.

Suppose $\alpha = 24$. We immediately have $257 \cdot 673 \| N$, hence $337 \| N$, so $13^2 | N$. To avoid having N u-def, the smallest component must be 3, 5, or 7.

If the smallest component is 7, then $97^2 \| N$ or else $97 \| N$ and $7^2 | N$. Therefore, $941 \| N$, so $193 \| N$. Then $3^2 11 \cdot 17 \| N$ or N is u-def. But $3^3 | \sigma^*(17 \cdot 257)$, so $3^3 | N$, a contradiction. Thus, the smallest component is not 7.

If the smallest component is 3, there are no more components $\equiv -1 \pmod 3$ as $3 | \sigma^*(257)$. Then we must have 7, 19, 25, and 31 as components or N is u-def. But then, no more than nine more odd components are allowable, and N is u-def. Therefore, the smallest component must be 5.

Because $5 \| N$, we must have $43 \| N$, since $5^2 | \sigma^*(43^2)$. We know that $13^2 | N$, so either $13^2 \| N$ or $13^3 \| N$ or $13^4 \| N$.

Suppose $13^4 \| N$. We cannot have 5^2 or 5^6 as components, so we must have 181 and 17^3. Starting with $2^{24}5 \| N$, we have, successively, as unitary divisors, $257 \cdot 673$, $337 \cdot 43$, 13^4, $17^3 181 \cdot 14281$, and $19^2 193$. Because $19^2 \| N$, we must have $3^9 37 \| N$. But $37^2 | \sigma^*(3^9 13281)$, contradicting $37 \| N$. Therefore, 13^4 is not a component.

Suppose $13^3 \| N$. Then $157 \| N$ or else $157^2 \| N$, hence $5^2 | N$. Consequently, $79 \| N$ and no more components $\equiv -1 \pmod 5$ are allowable. Then $97 \| N$ or else $97^2 \| N$, hence $5^2 | N$. If $7^3 \| N$, then $43^2 | N$, which cannot be, and if $7^4 \| N$, then $1201 \| N$, so $601 \| N$, and again $43^2 | N$. Therefore, $7^5 \| N$, so $191 \| N$. But then N is u-def.

Hence, $13^2 \| N$, so no more components $\equiv -1 \pmod 5$ are allowable. In particular, we must have $97 \| N$ to avoid $97^2 \| N$, and then we must have $3^3 7^3 | N$. But then N is u-def, so $\alpha = 24$ is impossible.

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