A PARTIAL ASYMPTOTIC FORMULA FOR THE NIVEN NUMBERS

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A Niven number is a positive integer that is divisible by its digital sum. That is, if n is an integer and $s(n)$ denotes the digital sum of n, then n is a Niven number if and only if $s(n)$ is a factor of n. This idea was introduced in [1] and investigated further in [2], [3], and [4].

One of the questions about the set N of Niven numbers was the status of

$$\lim_{x \to \infty} \frac{N(x)}{x},$$

where $N(x)$ denotes the number of Niven numbers less than x. This limit, if it exists, is called the "natural density" of N.

It was proven in [3] that the natural density of the set of Niven numbers is zero, and in [4] a search for an asymptotic formula for $N(x)$ was undertaken. That is, does there exist a function $f(x)$ such that

$$\lim_{x \to \infty} \frac{N(x)}{f(x)} = 1?$$

If such an $f(x)$ exists, then this would be indicated by the notation

$$N(x) \sim f(x).$$

Let k be a positive integer. Then k may be written in the form

$$k = 2^a 5^b t,$$

where $(t, 10) = 1$. In [4] the following notation was used.

$$N_k = \text{The set of Niven numbers with digital sum } k.$$
$$\overline{e}(k) = \text{The maximum of } a \text{ and } b. \qquad (1)$$
$$e(k) = \text{The order of 10 mod } t.$$

With this notation, it was then proven [4; Corollary 4.1] that

$$N_k(x) \sim c(\log x)^k, \qquad (2)$$

where c depends on k.

Thus, a partial answer concerning an asymptotic formula for $N(x)$ was found in [4]. Exact values of the constant c can be calculated for a given k. But, as noted in [4], this would involve an investigation of the partitions of k and solutions to certain Diophantine congruences. In what follows, we give the exact value of the constant c for a given integer k.

Let k be a positive integer such that $(k, 10) = 1$. We define the sets S and $\overline{S}$ as

$$S = \left\{ \langle x_i \rangle : \sum_{i=1}^{e(k)} x_i = k \right\},$$

and

$$\overline{S} = \left\{ \langle x_i \rangle : \sum_{i=1}^{e(k)} x_i = k \text{ and } \sum_{i=1}^{e(k)} 10^{i-1} x_i \equiv 0 \pmod{k} \right\},$$

where $\langle x_i \rangle$ is an $e(k)$-tuple of nonnegative integers. Since $(k, 10) = 1$, it follows that, for a positive integer n,

$$N_k(10^{e(k)n}) = \sum_{\langle x_i \rangle \in \overline{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10},$$

where $\binom{n}{t}_{10}$ denotes the t^{th} coefficient in the expansion of

$$G(x) = (1 + x + x^2 + \cdots + x^9)^n.$$

That is,

$$\frac{G^{(t)}(0)}{t!} = \binom{n}{t}_{10}, \tag{4}$$

where $G^{(t)}(0)$ is the t^{th} derivative of $G(x)$ at $x = 0$.

The expression given in (3) can be realized by noting that, for each

$$\langle x_i \rangle \in \overline{S},$$

the product

$$\prod_{i=1}^{e(k)} \binom{n}{x_i}_{10}$$

is the number of Niven numbers y less than $10^{e(k)n}$ with decimal representation

$$y = \sum_{j=1}^{ne(k)} y_j 10^{j-1}$$

such that

$$x_i = \sum_{j \equiv i \pmod{e(k)}} y_j.$$

Noting that $G^{(t)}(0) \sim n^t$, and using (4), we have that

$$\binom{n}{t}_{10} \sim \frac{n^t}{t!}.$$

Hence, for a positive k such that $(k, 10) = 1$, it follows from (3) that

$$N_k(10^{ne(k)}) \sim n^k \sum_{\langle x_i \rangle \in \overline{S}} \frac{1}{x_1! x_2! \ldots x_{e(k)}!}.$$

Therefore,

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!} \sum_{\langle x_i \rangle \in \overline{S}} \frac{k!}{x_1! x_2! \ldots x_{e(k)}!},$$

which may be rewritten in terms of multinomial coefficients as:

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!} \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\, x_2,\, \ldots,\, x_{e(k)}}. \tag{5}$$

Let w be the k^{th} root of unity $\exp(2\pi i/k)$, and consider the sum

$$\sum_{g=0}^{k-1} f(w^g),$$

where f is the function given by

$$f(u) = (u + u^{10} + u^{10^2} + \cdots + u^{10^{e(k)-1}})^k. \tag{6}$$

Then

$$\begin{aligned}\sum_{g=0}^{k-1} f(w^g) &= \sum_{g=0}^{k-1} \left(\sum_{i=0}^{e(k)-1} (w^g)^{10^i} \right)^k \\ &= \sum_{g=0}^{k-1} \sum_{\langle x_i \rangle \in S} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} (w^g)^{x_1 + 10x_2 + \cdots + 10^{e(k)-1}x_{e(k)}}\end{aligned} \tag{7}$$

In order to make the notation more compact, we will let

$$W(g, \langle x_i \rangle) = (w^g)^{x_1 + 10x_2 + \cdots + 10^{e(k)-1}x_{e(k)}}$$

Thus, after interchanging the order of summation, (7) becomes:

$$\begin{aligned}&\sum_{\langle x_i \rangle \in S} \sum_{g=0}^{k-1} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} W(g, \langle x_i \rangle) \\ &= \sum_{\langle x_i \rangle \in \overline{S}} \sum_{g=0}^{k-1} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} W(g, \langle x_i \rangle) \\ &\qquad + \sum_{\langle x_i \rangle \in S - \overline{S}} \sum_{g=0}^{k-1} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} W(g, \langle x_i \rangle) \\ &= \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} \sum_{g=0}^{k-1} W(g, \langle x_i \rangle) \\ &\qquad + \sum_{\langle x_i \rangle \in S - \overline{S}} \binom{k}{x_1,\, \ldots,\, x_{e(k)}} \sum_{g=0}^{k-1} W(g, \langle x_i \rangle).\end{aligned}$$

But noting that $W(g, \langle x_i \rangle)$ is equal to 1 when $\langle x_i \rangle \in \overline{S}$ and $\sum_{g=0}^{k-1} W(g, \langle x_i \rangle) = 0$ when $\langle x_i \rangle \in S - \overline{S}$, we conclude that

$$\sum_{g=0}^{k-1} f(w^g) = k \sum_{\langle x_i \rangle \in \overline{S}} \binom{k}{x_1,\, \ldots,\, x_{e(k)}}.$$

Hence, from (5), the following theorem is immediate.

Theorem 1: For any positive integer k, relatively prime to 10, let f, w, and $e(k)$ be given as above. Then

$$N_k(10^{ne(k)}) \sim \frac{n^k}{k!k} \sum_{g=0}^{k-1} f(w^g),$$

where n is any positive integer.

Some specific examples using Theorem 1 are:

$$N_3(10^n) \sim \frac{n^3}{6},$$

$$N_7(10^{6n}) \sim \frac{n^7}{7!7}(6^7 - 6),$$

$$N_{49}(10^{42n}) \sim \frac{n^{49}}{49!49}(42^{49} - 6(7^{49})),$$

and

$$N_{31}(10^{15n}) \sim \frac{n^{31}}{31!31}\left[15^{31} + 15\left(\left(\frac{-1 + (31)^{1/2}i}{2}\right)^{31} + \left(\frac{-1 - (31)^{1/2}i}{2}\right)^{31}\right)\right],$$

where $e(k) = 1, 6, 42$, and 15 when $k = 3, 7, 49$, and 31, respectively. Note that i denotes the square root of -1 in the last formula.

It is perhaps clear that the determination of such asymptotic formulas involves sums of complex expressions dependent on the orbit of 10 modulo k, and might be difficult to generalize.

Finally, we can use the above development as a model to generalize to the case where k is any positive integer, not necessarily relatively prime to 10. Recalling (1), we see that, if $(k, 10) \neq 1$, then it follows that $\bar{e}(k) \neq 0$. So $\bar{S}$ would be replaced by

$$\bar{S} = \left\{ \langle x_i; y_i \rangle : \sum_{i=1}^{e(k)} x_i + \sum_{i=1}^{\bar{e}(k)} y_i = k \right.$$

and

$$\left. \sum_{i=1}^{e(k)} x_i 10^{i+\bar{e}(k)-1} + \sum_{i=1}^{\bar{e}(k)} y_i 10^{i-1} \equiv 0 \pmod{k} \right\},$$

where y_i is a decimal digit for each i and where $\langle x_i; y_i \rangle$ is the $(e(k) + \bar{e}(k))$-tuple

$$(x_1, x_2, \ldots, x_{e(k)}, y_1, \ldots, y_{\bar{e}(k)}).$$

Thus, similarly to (3), it follows that

$$N_k(10^{ne(k)+\bar{e}(k)}) = \sum_{\langle x_i; y_i \rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10} \prod_{i=1}^{\bar{e}(k)} \binom{1}{y_i}_{10}. \tag{8}$$

But $\binom{1}{y_i}_{10} = 1$ for each $1 \leqslant i \leqslant \bar{e}(k)$, so (8) may be rewritten as

$$N_k(10^{ne(k)+\bar{e}(k)}) = \sum_{\langle x_i; y_i \rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10}.$$

Therefore,

$$N_k(10^{ne(k)+\bar{e}(k)}) \sim \sum_{\langle x_i; 0 \rangle \in \bar{S}} \prod_{i=1}^{e(k)} \binom{n}{x_i}_{10},$$

and replacing f as given in (6) by

$$f(u) = (u^{\bar{e}(k)} + \cdots + u^{\bar{e}(k)+e(k)-1})^k,$$

we are able to state the following theorem.

Theorem 2: For any positive integer k, let f, w, $e(k)$, and $\overline{e}(k)$ be given as above. Then

$$N_k(10^{ne(k)+\overline{e}(k)}) \sim \frac{n^k}{k!k}\sum_{g=0}^{k-1} f(w^g),$$

where n is any positive integer.

If $e(k) = 1$, the following corollary is also immediate since $f(w^g) = 1$ for each $0 \leqslant g \leqslant k - 1$.

Corollary: If k is a positive integer such that $e(k) = 1$, then, for any positive integer n,

$$N_k(10^{n+\overline{e}(k)}) \sim \frac{n^k}{k!}.$$

Using Theorem 2, we can determine an asymptotic formula for $N_k(x)$ for any positive real number x. This follows since there exists an integer n such that

$$10^{ne(k)+\overline{e}(k)} \leqslant x < 10^{(n+1)e(k)+\overline{e}(k)}. \tag{9}$$

But, by Theorem 2, we have that

$$N_k(10^{ne(k)+\overline{e}(k)}) \sim N_k(10^{(n+1)e(k)+\overline{e}(k)})$$

since $n_k \sim (n + 1)^k$. Hence,

$$N_k(x) \sim \frac{n^k}{k!k}\sum_{g=0}^{k-1} f(w^g),$$

and because (9) implies that

$$n \sim \frac{[\log x] - \overline{e}(k)}{e(k)} \sim \frac{\log x}{e(k)},$$

we have, in conclusion, Theorem 3.

Theorem 3: For any positive real number x and any positive integer k, let f, w, and $e(k)$ be given as above. Then

$$N_k(x) \sim \frac{(\log x)^k}{k!k(e(k))^k}\sum_{g=0}^{k-1} f(w^g).$$

Thus, an explicit formula for the constant c referred to in (2) has been given. The determination of an asymptotic formula for $N(x)$, however, is left as an open problem.

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