A MATRIX APPROACH TO CERTAIN IDENTITIES

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1. INTRODUCTION AND GENERALITIES

In the theory of functions of matrices [3], the domain of an analytic function f is extended to include a square matrix M of arbitrary order k by defining $f(M)$ as a polynomial in M of degree less than or equal to $k - 1$ provided f is defined on the spectrum of M. Then, if f is represented by a power series expansion in a circle containing the eigenvalues of M, this expansion remains valid when the scalar argument is replaced by the matrix M. Moreover, we point out that identities between functions of a scalar variable extend to matrix values of the argument. Thus, for example, the sum $(\sin M)^2 + (\cos M)^2$ equals the identity matrix of order k.

The purpose of this article is to use functions of two-by-two matrices Q to obtain a large number of Fibonacci-type identities, most of which we believe to be new.

To achieve this objective we generally proceed in the following way:

First we determine a closed form expression of the entries a_{ij} of any function $f(Q) = A = [a_{ij}]$ based on a polynomial representation of the function itself.

Then we consider a set of functions f such that $f(Q)$ can be found by means of a power series expansion $\hat{A} = [\hat{a}_{ij}] = f(Q)$ and equate a_{ij} and $\hat{a}_{ij}$ for some i and j, thus getting one or more Fibonacci-type identities.

We shall only be concerned with some of the elementary functions, namely, the square root function, the inverse function, and the exponential, circular, hyperbolic, and logarithm functions.

To illustrate the principles being used, we choose to proceed from the particular to the general, i.e., from use of the matrix Q defined in (1.3) to use of the more general matrix P defined in (2.7).

Throughout, we shall follow the usual notational convention that F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively.

First we recall ([2], [3]) that, if M has m distinct eigenvalues μ_k $(k = 1, 2, \ldots, m)$, the coefficients c_i of the polynomial representation

$$f(M) = \sum_{i=0}^{m-1} c_i M^i \tag{1.1}$$

of any analytic function f defined on the spectrum of M are given by the solution of the following system of m equations and m unknowns

$$\sum_{i=0}^{m-1} c_i \mu_k^i = f(\mu_k) \qquad (k = 1, 2, \ldots, m). \tag{1.2}$$

Then we consider the well-known matrix (e.g., see [4])

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \tag{1.3}$$

Since the distinct eigenvalues of Q are $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$, it follows from (1.1) and (1.2) that the coefficients c_0 and c_1 of the polynomial representation

$$f(Q) = c_0 I + c_1 Q \tag{1.4}$$

(where I denotes the two-by-two identity matrix)

of any function f defined on the spectrum of Q are given by the solution of the system

$$\begin{cases} c_0 + c_1\alpha = f(\alpha) \\ c_0 + c_1\beta = f(\beta). \end{cases} \tag{1.5}$$

In fact, from (1.5), we obtain

$$\begin{cases} c_0 = (\alpha f(\beta) - \beta f(\alpha))/\sqrt{5} \\ c_1 = (f(\alpha) - f(\beta))/\sqrt{5}. \end{cases} \tag{1.6}$$

Therefore, from (1.4) and (1.6), we can write

$$f(Q) = A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha f(\alpha) - \beta f(\beta) & f(\alpha) - f(\beta) \\ f(\alpha) - f(\beta) & \alpha f(\beta) - \beta f(\alpha) \end{bmatrix}. \tag{1.7}$$

It can be noted that the main property of the matrix Q, that is,

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \tag{1.8}$$

can be derived immediately from (1.7) by specializing f to the integral n^{th} power.

2. THE SQUARE ROOT MATRIX

In general, a two-by-two matrix possesses at least two square roots [3]. In the case of Q, the existence of a negative eigenvalue (β) implies that the entries a_{ij} of any square root A will be complex. Specializing f to the square root, from (1.7) we obtain the following equations defining one square root of Q,

$$\begin{cases} a_{11} = (\alpha\sqrt{\alpha} + i\sqrt{1/\alpha^3})/\sqrt{5} \\ a_{12} = a_{21} = (\sqrt{\alpha} - i\sqrt{1/\alpha})/\sqrt{5} \\ a_{22} = (\sqrt{1/\alpha} + i\sqrt{\alpha})/\sqrt{5}, \end{cases} \tag{2.1}$$

where $i = \sqrt{-1}$.

An alternative way to obtain a square root of Q is to solve the matrix equation $\hat{A}^2 = Q$, that is,

$$\begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2.2}$$

from which the following system can be written:

$$\begin{cases} \hat{a}_{11}^2 + \hat{a}_{12}\hat{a}_{21} = 1 \\ \hat{a}_{11}\hat{a}_{12} + \hat{a}_{12}\hat{a}_{22} = 1 \\ \hat{a}_{21}\hat{a}_{11} + \hat{a}_{22}\hat{a}_{21} = 1 \\ \hat{a}_{21}\hat{a}_{12} + \hat{a}_{22}^2 = 0. \end{cases} \tag{2.3}$$

From the second and third equations we can write

$$\hat{a}_{12}(\hat{a}_{11} + \hat{a}_{22}) = \hat{a}_{21}(\hat{a}_{11} + \hat{a}_{22}),$$

from which the equality $\hat{a}_{12} = \hat{a}_{21}$ is obtained (i.e., as expected, $\sqrt{Q}$ is a symmetric matrix). Therefore, from the fourth equation we get $\hat{a}_{12} = \hat{a}_{21} = \pm i\hat{a}_{22}$. Substituting these values in the first and second equations and dividing the corresponding sides one by the other, we obtain $\hat{a}_{11} = (1 \pm i)\hat{a}_{22}$. Hence, the solutions of the system (2.3) are:

$$\begin{cases} \hat{a}_{11} = (1 \pm i)\hat{a}_{22} \\ \hat{a}_{12} = \hat{a}_{21} = \pm i\hat{a}_{22} \\ \hat{a}_{22} = \pm\sqrt{(-1 \mp 2i)/5}. \end{cases} \tag{2.4}$$

Since

$$-1 \mp 2i = \sqrt{5}\, e^{i(\pi \pm \arctan 2)},$$

the complex entry $\hat{a}_{22}$ can be written as

$$\hat{a}_{22} = (1/5)^{1/4}\, e^{i(\pi \pm \arctan 2)/2 + ik\pi} \qquad (k = 0, 1).$$

The real part of $\hat{a}_{22}$ is

$$\operatorname{Re}(\hat{a}_{22}) = (-1)^k (1/5)^{1/4} \cos \frac{\pi \pm \arctan 2}{2} \quad (k = 0,\ 1). \tag{2.5}$$

Since every square root of Q must satisfy (2.3), the matrix A defined by (2.1) does. Equating the real parts of a_{22} and $\hat{a}_{22}$, and squaring both sides of this equation, from (2.1) and (2.5) we have

$$1/(5\alpha) = \sqrt{1/5} \sin^2 \frac{\arctan 2}{2},$$

thus obtaining the trigonometrical identity

$$\alpha = 1/\left(\sqrt{5} \sin^2 \frac{\arctan 2}{2}\right). \tag{2.6}$$

Equating the imaginary parts of a_{22} and $\hat{a}_{22}$, we obtain the equivalent identity

$$\alpha = \sqrt{5} \cos^2 \frac{\arctan 2}{2}. \tag{2.6$'$}$$

The preceding treatment may be generalized in the following way:

Let

$$P = \begin{bmatrix} p & 1 \\ 1 & 0 \end{bmatrix} \tag{2.7}$$

whence, by induction

$$P^n = \begin{bmatrix} U_{n+1} & U_n \\ U_n & U_{n-1} \end{bmatrix} \tag{2.8}$$

where U_n $(n = 0, 1, 2, \ldots)$ is defined by the recurrence relation

$$U_{n+2} = pU_{n+1} + U_n;\ U_0 = 0,\ U_1 = 1. \tag{2.9}$$

When $p = 1$, we get the Fibonacci numbers F_n. When $p = 2$, the Pell numbers P_n result.

Writing

$$\Delta = \sqrt{p^2 + 4}, \tag{2.10}$$

we find that the eigenvalues of P in (2.7) are

$$\alpha_p = (p + \Delta)/2,\ \beta_p = (p - \Delta)/2. \tag{2.11}$$

From (2.11) and (2.10), it can be noted that $\alpha_p\beta_p = -1$, i.e., $\beta_p = -1/\alpha_p$.

When $p = 1$, these eigenvalues are $(1 \pm \sqrt{5})/2$ as given earlier (namely, the values of $\alpha = \alpha_1$ and $\beta = \beta_1$). If $p = 2$, these eigenvalues reduce to

$$\alpha_2 = 1 + \sqrt{2} \quad \text{and} \quad \beta_2 = 1 - \sqrt{2}.$$

Paralleling the argument for Fibonacci numbers outlined above, we may derive the identity corresponding to (2.6):

$$\alpha_p = 1/\left(\Delta \sin^2 \frac{\arctan(2/p)}{2}\right). \tag{2.12}$$

Taking $p = 2$, we have the identity for Pell numbers:

$$\alpha_2 = 1/\left(2\sqrt{2}\,\sin^2\frac{\arctan 1}{2}\right). \tag{2.13}$$

It must be noted that identity (2.12) may be verified directly. In fact, the identity $\sin^2(x/2) = (1 - \cos x)/2$ implies

$$\sin^2\frac{\arctan(2/p)}{2} = \frac{1 - \cos(\arctan(2/p))}{2} = (1 - p/\sqrt{p^2 + 4})/2$$
$$= (1 - p/\Delta)/2 = (\Delta - p)/(2\Delta) = -\beta_p/\Delta = 1/(\alpha_p\Delta).$$

3. THE EXPONENTIAL FUNCTION MATRIX

The previous results follow for $f(x) = \sqrt{x}$. Other particular identities emerge for other choices of f. Specializing f to the exponential function, from (1.7) we obtain:

$$\begin{cases} a_{11} = (\alpha e^{\alpha} - \beta e^{\beta})/\sqrt{5} \\ a_{12} = a_{21} = (e^{\alpha} - e^{\beta})/\sqrt{5} \\ a_{22} = (\alpha e^{\beta} - \beta e^{\alpha})/\sqrt{5}. \end{cases} \tag{3.1}$$

An alternative way of obtaining $\hat{A} = [\hat{a}_{ij}] = \exp Q$ is (see [1], [5], [6]) to use the power series expansion

$$\exp Q = \sum_{n=0}^{\infty}\frac{Q^n}{n!}. \tag{3.2}$$

From (1.8), it is easily seen that:

$$\begin{cases} \hat{a}_{11} = \sum_{n=0}^{\infty}\frac{F_{n+1}}{n!} \\ \hat{a}_{12} = \hat{a}_{21} = \sum_{n=0}^{\infty}\frac{F_n}{n!} \\ \hat{a}_{22} = \sum_{n=0}^{\infty}\frac{F_{n-1}}{n!}. \end{cases} \tag{3.3}$$

Therefore, equating the corresponding entries of $\hat{A}$ and A, from (3.1) and (3.3) we obtain the following known Fibonacci identities (see [4]):

$$\sum_{n=0}^{\infty}\frac{F_n}{n!} = (e^{\alpha} - e^{\beta})/\sqrt{5} \tag{3.4}$$

$$\sum_{n=0}^{\infty}\frac{F_{n+1}}{n!} = (\alpha e^{\alpha} - \beta e^{\beta})/\sqrt{5} \tag{3.5}$$

$$\sum_{n=0}^{\infty}\frac{F_{n-1}}{n!} = (\alpha e^{\beta} - \beta e^{\alpha})/\sqrt{5}. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\sum_{n=0}^{\infty} \frac{L_n}{n!} = e^{\alpha} + e^{\beta}. \tag{3.7}$$

It is evident that the above results may be generalized by using the exponential of the matrix P. As an example, for $p = 2$, the following identity involving Pell numbers,

$$\sum_{n=0}^{\infty} \frac{P_n}{n!} = e(e^{\sqrt{2}} - e^{-\sqrt{2}})/(2\sqrt{2}), \tag{3.8}$$

is obtained. Similar results to those in (3.5)-(3.7) readily follow.

4. OTHER FUNCTIONAL MATRICES

Let us consider the following power series expansions ([3], [6]):

$$\sin Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n+1}}{(2n+1)!} \tag{4.1}$$

$$\cos Q = \sum_{n=0}^{\infty} (-1)^n \frac{Q^{2n}}{(2n)!} \tag{4.2}$$

$$\sinh Q = \sum_{n=0}^{\infty} \frac{Q^{2n+1}}{(2n+1)!} \tag{4.3}$$

$$\cosh Q = \sum_{n=0}^{\infty} \frac{Q^{2n}}{(2n)!}. \tag{4.4}$$

Using reasoning similar to the preceding, we may obtain a large number of Fibonacci identities, some of which are well known [6]. These identities have the following general forms,

$$\sum_{n=0}^{\infty} c_n F_n = (f(\alpha) - f(\beta))/\sqrt{5}, \tag{4.5}$$

$$\sum_{n=0}^{\infty} c_n F_{n+1} = (\alpha f(\alpha) - \beta f(\beta))/\sqrt{5}, \tag{4.6}$$

$$\sum_{n=0}^{\infty} c_n F_{n-1} = (\alpha f(\beta) - \beta f(\alpha))/\sqrt{5}, \tag{4.7}$$

where

$$f(y) = \sum_{n=0}^{\infty} c_n y^n.$$

A brief selection of particular cases is shown below:

$$\sum_{n=0}^{\infty} (-1)^n \frac{F_{2n+1}}{(2n+1)!} = (\sin\alpha - \sin\beta)/\sqrt{5} \tag{4.8}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{F_{2n}}{(2n)!} = (\cos\alpha - \cos\beta)/\sqrt{5} \tag{4.9}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n+1)!} = (\sinh \alpha - \sinh \beta)/\sqrt{5} \tag{4.10}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n}}{(2n)!} = (\cosh \alpha - \cosh \beta)/\sqrt{5} \tag{4.11}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n+1}}{(2n)!} = (\alpha \cosh \alpha - \beta \cosh \beta)/\sqrt{5} \tag{4.12}$$

$$\sum_{n=0}^{\infty} \frac{F_{2n-1}}{(2n)!} = (\alpha \cosh \beta - \beta \cosh \alpha)/\sqrt{5}. \tag{4.13}$$

Combining some of the above-mentioned results, we may obtain analogous identities involving Lucas numbers. For example, combining (4.12) and (4.13) gives

$$\sum_{n=0}^{\infty} \frac{L_{2n}}{(2n)!} = \cosh \alpha + \cosh \beta. \tag{4.14}$$

Again, we point out that these identities may be generalized by using circular and hyperbolic functions of the matrix P. In particular, we may obtain results for Pell numbers similar to these listed for Fibonacci and Lucas numbers.

5. EXTENSIONS

The results obtained *primo impetu* in Sections 3 and 4 may be extended using functions of the matrix

$$Q_{k,x} = xQ^k = \begin{bmatrix} xF_{k+1} & xF_k \\ xF_k & xF_{k-1} \end{bmatrix}, \tag{5.1}$$

where x is an arbitrary real quantity and k is a nonnegative integer. Since $Q_{k,x}$ is a polynomial $r(Q)$ in Q, it follows that its eigenvalues are

$$\begin{cases} \chi_1(k, x) = r(\alpha) = x\alpha^k \\ \chi_2(k, x) = r(\beta) = x\beta^k, \end{cases} \tag{5.2}$$

and $f(Q_{k,x}) = f(r(Q))$ derives values in terms of $f(r(\alpha))$ and $f(r(\beta))$. Thus, any function f defined on the spectrum of $Q_{k,x}$ can be obtained from (1.7) by replacing $f(\alpha)$ and $f(\beta)$ with $f(\chi_1(k, x))$ and $f(\chi_2(k, x))$, respectively. Moreover, from (5.1) and (1.8), it is easily seen that $Q_{k,x}$ enjoys the property

$$Q_{k,x}^n = (xQ^k)^n = x^nQ^{kn} = \begin{bmatrix} x^nF_{kn+1} & x^nF_{kn} \\ x^nF_{kn} & x^nF_{kn-1} \end{bmatrix}. \tag{5.3}$$

5.1 The Exponential Function of $Q_{k,x}$

Specializing f to the exponential function, from (1.7) and (5.2) we obtain the following values of the entries of the polynomial representation $A_{k,x} = [a_{ij}(k, x)]$ of $\exp Q_{k,x}$:

$$\begin{cases} a_{11}(k, x) = (\alpha e^{x\alpha^k} - \beta e^{x\beta^k})/\sqrt{5} \\ a_{12}(k, x) = a_{21}(k, x) = (e^{x\alpha^k} - e^{x\beta^k})/\sqrt{5} \\ a_{22}(k, x) = (\alpha e^{x\beta^k} - \beta e^{x\alpha^k})/\sqrt{5}. \end{cases} \tag{5.4}$$

Calculating $\exp Q_{k,x}$ by means of (3.2), we have

$$\exp Q_{k,x} = \sum_{n=0}^{\infty} \frac{Q^n_{k,x}}{n!} = \hat{A}_{k,x} = [\hat{a}_{ij}(k, x)]. \tag{5.5}$$

Equating $\hat{a}_{ij}(k, x)$ and $a_{ij}(k, x)$, from (5.5), (5.3), and (5.4) we obtain:

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn+1}}{n!} = (\alpha e^{x\alpha^k} - \beta e^{x\beta^k})/\sqrt{5} \tag{5.6}$$

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn}}{n!} = (e^{x\alpha^k} - e^{x\beta^k})/\sqrt{5} \tag{5.7}$$

$$\sum_{n=0}^{\infty} \frac{x^n F_{kn-1}}{n!} = (\alpha e^{x\beta^k} - \beta e^{x\alpha^k})/\sqrt{5}. \tag{5.8}$$

Combining (5.6) and (5.8), we get

$$\sum_{n=0}^{\infty} \frac{x^n L_{kn}}{n!} = e^{x\alpha^k} + e^{x\beta^k}. \tag{5.9}$$

The above results (5.6)-(5.9) may be generalized using the exponential of the matrix xP^k [refer to (2.8)].

5.2 Circular and Hyperbolic Functions of $Q_{k,x}$

By means of a procedure similar to the preceding one, the use of $\sin Q_{k,x}$, $\cos Q_{k,x}$, $\sinh Q_{k,x}$, and $\cosh Q_{k,x}$ yields a set of identities having the following general forms,

$$\sum_{n=0}^{\infty} c_n x^n F_{kn} = (f(x\alpha^k) - f(x\beta^k))/\sqrt{5}, \tag{5.10}$$

$$\sum_{n=0}^{\infty} c_n x^n F_{kn+1} = (\alpha f(x\alpha^k) - \beta f(x\beta^k))/\sqrt{5}, \tag{5.11}$$

$$\sum_{n=0}^{\infty} c_n x^n F_{kn-1} = (\alpha f(x\beta^k) - \beta f(x\alpha^k))/\sqrt{5}, \tag{5.12}$$

where

$$f(y) = \sum_{n=0}^{\infty} c_n y^n.$$

A brief selection of particular cases is shown below:

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} F_{k(2n+1)}}{(2n+1)!} = (\sin(x\alpha^k) - \sin(x\beta^k))/\sqrt{5} \tag{5.13}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} F_{2kn}}{(2n)!} = (\cos(x\alpha^k) - \cos(x\beta^k))/\sqrt{5} \tag{5.14}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n+1} F_{k(2n+1)}}{(2n+1)!} = (\sinh(x\alpha^k) - \sinh(x\beta^k))/\sqrt{5} \tag{5.15}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} F_{2kn}}{(2n)!} = (\cosh(x\alpha^k) - \cosh(x\beta^k))/\sqrt{5} \tag{5.16}$$

$$\sum_{n=0}^{\infty} \frac{x^{2n} L_{2kn}}{(2n)!} = \cosh(x\alpha^k) + \cosh(x\beta^k). \tag{5.17}$$

The above-mentioned identities may be generalized using circular and hyperbolic functions of the matrix xP^k [refer to (2.8)].

5.3 The Logarithm of $Q_{k,x}$ for k Even and Particular Values of x

The principal value of the function $\ln Q$ can be calculated by (1.7), thus getting a complex matrix A. Unfortunately, since Q has a negative eigenvalue, the power series expansion of the matrix logarithm (see [3])

$$\ln Q = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(Q - I)^n \tag{5.18}$$

does not converge and a matrix $\hat{A}$ cannot be obtained in this way. On the other hand, the use of $Q_{k,x}$, with k even, allows us to utilize this function. We will show how, setting x equal to the reciprocal of the k^{th} Lucas number, some interesting results can be worked out.

First we define the two-by-two matrix

$$R_{k,x} = Q_{k,x} - I = xQ^k - I \tag{5.19}$$

whence, using induction, it can be proved that, if n is a nonnegative integer, then

$$R^n_{k,1/L_k} = \frac{1}{L^n_k}\begin{bmatrix} (-1)^n F_{kn-1} & (-1)^{n+1} F_{kn} \\ (-1)^{n+1} F_{kn} & (-1)^n F_{kn+1} \end{bmatrix}. \tag{5.20}$$

Incidentally, it can also be proved that

$$R^n_{2,1/2} = \frac{1}{2^n}\begin{bmatrix} (-1)^n F_{n-1} & (-1)^{n+1} F_n \\ (-1)^{n+1} F_n & (-1)^n F_{n+1} \end{bmatrix} \tag{5.21}$$

Then replacing f in (1.7) with the function $f(y) = \ln(xy^k)$, we have $f(\alpha) = \ln(x\alpha^k)$, $f(\beta) = \ln(x\beta^k)$, and we calculate the matrix

$$\ln Q_{k,x} = A_{k,x} = [a_{ij}(k, x)]$$

which is real if and only if k is even and $x > 0$. In fact, we obtain

$$\begin{cases} a_{11}(k, x) = \dfrac{k}{\sqrt{5}} \ln \alpha + \ln x \\[2ex] a_{12}(k, x) = a_{21}(k, x) = \dfrac{2k}{\sqrt{5}} \ln \alpha \\[2ex] a_{22}(k, x) = -\dfrac{k}{\sqrt{5}} \ln \alpha + \ln x \end{cases} \tag{5.22}$$

where it can be noted that $a_{12}(k, x) = a_{21}(k, x)$ is independent of x.

Finally, since for k even the inequality

$$|\chi_i(k, 1/L_k) - 1| < 1 \qquad (i = 1, 2)$$

holds [see (5.2)], we can calculate the function $\ln Q_{k,1/L_k}$ by means of the power series expansion (5.18):

$$\ln Q_{k,1/L_k} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} R^n_{k,1/L_k} = \hat{A}_{k,1/L_k} = [\hat{a}_{ij}(k, 1/L_k)]. \tag{5.23}$$

Replacing x by $1/L_k$ in (5.22) and equating $\hat{a}_{ij}(k, 1/L_k)$ and $a_{ij}(k, 1/L_k)$, from (5.23), (5.20), and (5.22), we obtain:

$$\sum_{n=1}^{\infty} \frac{F_{kn-1}}{nL_k^n} = \ln L_k - \frac{k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots) \tag{5.24}$$

$$\sum_{n=1}^{\infty} \frac{F_{kn}}{nL_k^n} = \frac{2k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots) \tag{5.25}$$

$$\sum_{n=1}^{\infty} \frac{F_{kn+1}}{nL_k^n} = \ln L_k + \frac{k}{\sqrt{5}} \ln \alpha \qquad (k = 0, 2, 4, \ldots). \tag{5.26}$$

Combining (5.24) and (5.26), we have

$$\sum_{n=1}^{\infty} \frac{L_{kn}}{nL_k^n} = \ln L_k^2 \qquad (k = 0, 2, 4, \ldots). \tag{5.27}$$

Using the matrix $Q_{2,1/2}$ [see (5.21)], by means of the same procedure we obtain

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n} = \frac{4}{\sqrt{5}} \ln \alpha = \sum_{n=1}^{\infty} \frac{F_{2n}}{n3^n} \tag{5.28}$$

and

$$\sum_{n=1}^{\infty} \frac{L_n}{n2^n} = \ln 4, \tag{5.29}$$

where the right-hand side of (5.28) was derived by setting $k = 2$ in (5.25).

We conclude this subsection by pointing out that, from the equality

$$(Q^k/L_k - I)^n = R^n_{k,\,1/L_k}$$

[directly derived from (5.19)] and from (5.20), the following identities can be obtained:

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\frac{F_{ki\pm 1}}{L_k^i} = (-1)^n\frac{F_{kn\mp 1}}{L_k^n} \tag{5.30}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\frac{F_{ki}}{L_k^i} = (-1)^{n+1}\frac{F_{kn}}{L_k^n} \tag{5.31}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}\frac{L_{ki}}{L_k^i} = (-1)^n\frac{L_{kn}}{L_k^n}. \tag{5.32}$$

5.4 The Inverse of $I - Q_{k,x}$

Let us consider the two-by-two matrix

$$S_{k,x} = -R_{k,x} = I - Q_{k,x} = I - xQ^k. \tag{5.33}$$

For

$$x \neq \begin{cases} \alpha^k,\ \beta^k & (k \text{ even}) \\ -\alpha^k,\ -\beta^k & (k \text{ odd}), \end{cases} \tag{5.34}$$

$S_{k,x}$ admits its inverse

$$S^{-1}_{k,x} = \frac{1}{D}\begin{bmatrix} 1 - xF_{k-1} & xF_k \\ xF_k & 1 - xF_{k+1} \end{bmatrix} = A_{k,x} = [a_{ij}(k,\ x)], \tag{5.35}$$

where

$$D = (-1)^k x^2 - xL_k + 1.$$

The inverse of $S_{k,x}$ can be obtained from (1.7) by replacing $f(\alpha)$ and $f(\beta)$ with $1/(1 - x\alpha^k)$ and $1/(1 - x\beta^k)$, respectively.

It is apparent that the inequality

$$|\chi_i(k,\ x)| < 1 \qquad (i = 1,\ 2)$$

holds for $-\alpha^{-k} < x < \alpha^{-k}$ [see (5.2)]. Under this restriction, we can calculate $S^{-1}_{k,x}$ by means of the power series expansion [3]:

$$S^{-1}_{k,x} = \sum_{n=0}^{\infty} Q^n_{k,x} = \hat{A}_{k,x} = [\hat{a}_{ij}(k,\ x)] \tag{5.36}$$

Equating $\hat{a}_{ij}(k,\ x)$ and $a_{ij}(k,\ x)$, from (5.36), (5.3), and (5.35), we obtain:

$$\sum_{n=0}^{\infty} x^n F_{kn+1} = (1 - xF_{k-1})/D \qquad (-\alpha^{-k} < x < \alpha^{-k}) \tag{5.37}$$

$$\sum_{n=0}^{\infty} x^n F_{kn} = xF_k/D \qquad (-\alpha^{-k} < x < \alpha^{-k}) \tag{5.38}$$

$$\sum_{n=0}^{\infty} x^n F_{kn-1} = (1 - xF_{k+1})/D \quad (-\alpha^{-k} < x < \alpha^{-k}). \tag{5.39}$$

Combining (5.37) and (5.39), we have

$$\sum_{n=0}^{\infty} x^n L_{kn} = (2 - xL_k)/D \quad (-\alpha^{-k} < x < \alpha^{-k}). \tag{5.40}$$

Setting $k = 1$ and $x = 1/2$ in (5.38), we obtain, as a particular case,

$$\sum_{n=0}^{\infty} \frac{F_n}{2^n} = 2. \tag{5.41}$$

Setting $k = 1$ and $x = 1/2$, $1/3$ in (5.40), we have

$$\sum_{n=0}^{\infty} \frac{L_n}{2^n} = 6, \tag{5.42}$$

and

$$\sum_{n=0}^{\infty} \frac{L_n}{3^n} = 3, \tag{5.43}$$

respectively.

6. CONCLUDING REMARKS

While the authors know that a few of the results presented in this article have been established by others (e.g., [1], [5], [6]), they believe that most of them are original. Certainly, more possibilities exist than those developed here.

It is possible that some of the work presented above could be extended to simple cases of three-by-three matrices.

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