ELLIPTIC FUNCTIONS AND LAMBERT SERIES IN THE SUMMATION OF RECIPROCALS IN CERTAIN RECURRENCE-GENERATED SEQUENCES

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1. INTRODUCTION

Consider the sequence of positive integers $\{w_n\}$ defined by the recurrence relation

$$w_{n+2} = pw_{n+1} - qw_n \tag{1.1}$$

with initial conditions

$$w_0 = a,\ w_1 = b, \tag{1.2}$$

where $a \geqslant 0$, $b \geqslant 1$, $p \geqslant 1$, $q \neq 0$ are integers with $p^2 \geqslant 4q$. We first consider the "nondegenerate" case: $p^2 > 4q$.

Roots of the characteristic equations of (1.1), namely,

$$\lambda^2 - p\lambda + q = 0 \tag{1.3}$$

are

$$\begin{cases} \alpha = (p + \sqrt{p^2 - 4q})/2, \\ \beta = (p - \sqrt{p^2 - 4q})/2. \end{cases} \tag{1.4}$$

Note $\alpha > 0$, $\beta \gtrless 0$ depending on $q \gtrless 0$. Then

$$\alpha + \beta = p,\ \alpha\beta = q,\ \alpha - \beta = \sqrt{p^2 - 4q} > 0. \tag{1.5}$$

The explicit *Binet form* for w_n is

$$w_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} \tag{1.6}$$

in which

$$\begin{cases} A = b - a\beta, \\ B = b - a\alpha. \end{cases} \tag{1.7}$$

It is the purpose of this paper to investigate the infinite sums

$$\sum_{n=1}^{\infty} \frac{1}{w_n} \tag{1.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n}} \tag{1.9}$$

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n-1}}. \tag{1.10}$$

Special cases of $\{w_n\}$ which interest us here are:

the *Fibonacci sequence* $\{F_n\}$: $a = 0$, $b = 1$, $p = 1$, $q = -1$; (1.11)

the *Lucas sequence* $\{L_n\}$: $a = 2$, $b = 1$, $p = 1$, $q = -1$; (1.12)

the *Pell sequence* $\{P_n\}$: $a = 0$, $b = 1$, $p = 2$, $q = -1$; (1.13)

the *Pell-Lucas sequence* $\{Q_n\}$: $a = 2$, $b = 2$, $p = 2$, $q = -1$; (1.14)

the *Fermat sequence* $\{f_n\}$: $a = 0$, $b = 1$, $p = 3$, $q = 2$; (1.15)

the *"Fermat-Lucas" sequence* $\{g_n\}$: $a = 2$, $b = 3$, $p = 3$, $q = 2$; (1.16)

the *generalized Fibonacci sequence* $\{U_n\}$: $a = 0$, $b = 1$; (1.17)

the *generalized Lucas sequence* $\{V_n\}$: $a = 2$, $b = p$. (1.18)

The Fermat sequence (1.15) is also known as the *Mersenne sequence.*

Binet forms and related information are readily deduced for (1.11)-(1.18) from (1.4)-(1.7). Notice that $f_n = 2^n - 1$, $g_n = 2^n + 1$, and, for both (1.15) and (1.16), $\alpha = 2$, $\beta = 1$, in which case the roots of the characteristic equation are not irrational.

Sequences (1.11), (1.13), (1,15), and (1.17), in which $a = 0$, $b = 1$, may be alluded to as being of *Fibonacci type.* On the other hand, sequences (1.12), (1.14), (1.16), and (1.18), in which $a = 2$, $b = p$, may be said to be of *Lucas type.*

For Fibonacci-type sequences, we have $A = B = 1$, and the Binet form (1.6) reduces to

$$w_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{1.6$'$}$$

whereas for Lucas-type sequences, in which $A = -B = \alpha - \beta$, we have the simpler form

$$w_n = \alpha^n + \beta^n. \tag{1.6$''$}$$

From (1.6),

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{\dfrac{1}{w_{n+1}}}{\dfrac{1}{w_n}}\right] = \lim_{n\to\infty}\frac{w_n}{w_{n+1}} &= \lim_{n\to\infty}\left[\frac{A\alpha^n - B\beta^n}{A\alpha^{n+1} - B\beta^{n+1}}\right] \\ &= \frac{1}{\alpha}\lim_{n\to\infty}\left[\frac{A - B\left(\dfrac{\beta}{\alpha}\right)^n}{A - B\left(\dfrac{\beta}{\alpha}\right)^{n+1}}\right] = \frac{1}{\alpha} \quad \text{since } \left|\frac{\beta}{\alpha}\right| < 1 \\ &< 1 \quad \text{since } \alpha > 1.\end{aligned} \tag{1.19}$$

To prove this last assertion, we note that $2\alpha = p + \sqrt{p^2 - 4q} \geqslant 1 + 1 = 2$. If $p + \sqrt{p^2 - 4q} = 2$, then $q = p - 1$; but $q \neq 0$, so $p \neq 1 \Rightarrow p > 1 \Rightarrow \alpha > 1$.

Thus,

$$\sum_{n=1}^{\infty} \frac{1}{w_n} \text{ converges absolutely.} \tag{1.20}$$

All the sequences (1.11)-(1.18) satisfy (1.20).

2. BACKGROUND

Historical

The desire to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \tag{2.1}$$

seems to have been stated first by Laisant [21] in 1899 in these words:

> "*A-t-on déjà étudié la série*
>
> $$\frac{1}{1}\ \frac{1}{1}\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{5} \ldots,$$
>
> *que forment les inverses des termes de Fibonacci, et qui est évidemment convergente?*"

Barriol [3] responded to this challenge by approximating (2.1) to 10 decimal places:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3.3598856662\ldots \tag{2.1$'$}$$

which concurs with that obtained by Brousseau ([6], p. 45) in calculating

$$\sum_{n=1}^{400} \frac{1}{F_n} \tag{2.1$''$}$$

to 400 decimal places. (Actually, in (2.1)$'$, the first decimal digit, 3, is misprinted in [3] as 2.) However, we find in Escott [11] the claim:

> "*J'ai calculé la valeur de cette somme avec quinze décimales et vérifié les résultats à l'aide de la formule*
>
> $$\frac{1}{p_{n+2}} = \frac{1}{p_n} - \frac{1}{p_{n+1}} - \frac{(-1)^n}{p_n p_{n+1} p_{n+2}}$$
>
> *où p_n est le $n^{\text{ième}}$ terme de la série de Fibonacci.*
>
> *J'obtiens* 3,3598856672—*qui diffère du résultat de M. Barriol par le* 10^e *chiffre.*"

For the Lucas numbers, the approximation corresponding to (2.1)$''$ given by Brousseau ([6], p. 45) is

$$\sum_{n=1}^{400} \frac{1}{L_n} = 1.9628581732\ldots \tag{2.2}$$

Catalan [9] in 1883, and earlier Lucas [24] in 1878, had divided the problem of investigating $\sum_{n=1}^{\infty}(1/F_n)$ into two parts, namely,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}}, \text{ expressible in terms of Jacobian elliptic functions,} \tag{2.3}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}}, \text{ expressible in terms of Lambert series.} \tag{2.4}$$

Landau [23] in 1899 elaborated on Catalan's result in the case of (2.3) by expressing the answer in terms of *theta functions*.

Moreover, Catalan [9] also obtained an expression for

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} \tag{2.5}$$

in terms of Jacobian elliptic functions. No mention in the literature available to me was made by Catalan for

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}}. \tag{2.6}$$

Results for Pell and Pell-Lucas numbers corresponding to those in (2.3)-(2.6) were obtained in [26] by Horadam and Mahon.

For a wealth of detailed, numerical information on the matters contained in, and related to, (2.3)-(2.6), one might consult Bruckman [7], who obtained closed forms for the expressions in (2.3) and (2.5), among others, in terms of certain constants defined by Jacobian elliptic functions.

Observe in passing that in (2.5) the value $n = 0$ is omitted in the summation even though $L_0 = 2$ $(\neq 0)$. We do this for consistency because, in the non-Lucas type sequences, $a = 0$ (i.e., $w_0 = 0$, so $1/w_0$ is infinite).

From (1.6),

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{w_n} &= (\alpha - \beta) \sum_{n=1}^{\infty} \frac{1}{A\alpha^n - B\beta^n} = (\alpha - \beta) \sum_{n=1}^{\infty} \frac{\beta^n}{A\alpha^n\beta^n - B\beta^{2n}} \qquad (2.7)\\ &= (\alpha - \beta) \sum_{n=1}^{\infty} \frac{\beta^n}{Aq^n - B\beta^{2n}} \quad \text{by (1.5)}\\ &= (\alpha - \beta) \sum_{n=1}^{\infty} \frac{(1/A)\beta^n}{q^n - (B/A)\beta^{2n}}.\end{aligned}$$

At this stage, we must pause. The algebra, it appears, is too fragile to bear the burden of both q^n and B/A being simultaneously unrestricted, so some constraints must be imposed.

Clearly, the evenness or oddness of n is important since q^n will alternate in sign if $q < 0$. Following historical precedent as indicated earlier, we find it necessary to dichotomize w_n into the cases n even, n odd.

Furthermore, the outcome of the expression on the right-hand side of (2.7) depends on whether B/A (or A/B) is >0 or <0.

For our purposes, two specific values concern us, viz.,

$$\frac{A}{B} = \pm 1.$$

I. $\frac{A}{B} = 1$

From (1.7), $A/B = 1$ means that

$$b - a\alpha = b - a\beta \qquad (\alpha \neq \beta),$$

whence

$$a = 0$$

without any new restrictions on b, p, or q. Combining this fact with the criterion for (1.6)′ (i.e., $b = 1$), we have

$$a = 0,\ b = 1 \Rightarrow A = B = 1. \tag{2.8}$$

Sequences satisfying the criteria $a = 0$, $b = 1$ are the Fibonacci-type sequences.

II. $\frac{A}{B} = -1$

In this case, (1.7) gives

$$\begin{aligned} b - a\alpha &= -(b - a\beta) \\ b &= \frac{ap}{2} && \text{by (1.5)} \\ &= p && \text{if } a = 2. \end{aligned}$$

Relating these criteria to (1.6)″, we see that

$$a = 2,\ b = p \Rightarrow A = -B = \alpha - \beta. \tag{2.9}$$

Sequences which satisfy the criteria $a = 2$, $b = p$ are the Lucas-type sequences.

Having set down some necessary background information, we now proceed to the main objective of the paper, to wit, the application to our summation requirements of Jacobian elliptic functions and Lambert series.

3. JACOBIAN ELLIPTIC FUNCTIONS

In *Jacobian elliptic function* theory, the elliptic integral constants (see [7], [18])

$$K = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} \tag{3.1}$$

and

$$K' = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k'^2 \sin^2 t}} \tag{3.2}$$

are related by

$$k^2 + k'^2 = 1, \tag{3.3}$$

k' being the *complement* of k.

Write

$$r = e^{-K'\pi/K} \qquad (0 < r < 1). \tag{3.4}$$

Jacobi's symbol q [17] is here replaced by r to avoid confusion with the use of q in the recurrence relation (1.1).

Two of Jacobi's summation formulas [18] required for our purposes are

$$\frac{2K}{\pi} = 1 + \frac{4r}{1 + r^2} + \frac{4r^2}{1 + r^4} + \frac{4r^3}{1 + r^6} + \cdots \tag{3.5}$$

and

$$\frac{2kK}{\pi} = \frac{4\sqrt{r}}{1 + r} + \frac{4\sqrt{r^3}}{1 + r^3} + \frac{4\sqrt{r^5}}{1 + r^5} + \cdots . \tag{3.6}$$

Now, from (1.6).

$$\begin{aligned}
\frac{1}{w_{2n-1}} &= \frac{\alpha - \beta}{A(\alpha^{2n-1} - (B/A)\beta^{2n-1})} && \qquad (3.7)\\
&= (\alpha - \beta)\cdot \frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} - \beta^{4n-2}} && \text{if } A = B = 1\\
&= (\alpha - \beta)\cdot \frac{\beta^{2n-2}\beta}{-1 - \beta^{4n-2}} && \text{if } q = -1 \text{ in (1.5)}\\
&= (\alpha - \beta)\cdot\sqrt{r}\cdot \frac{r^{n-1}}{1 + r^{2n-1}} && \text{with } \begin{cases} r = \beta^2 \ (\beta < 0) \\ \sqrt{r} = -\beta, \text{ so } 0 < \sqrt{r} < 1. \end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{w_{2n-1}} &= (\alpha - \beta)\cdot\sqrt{r}\sum_{n=1}^{\infty} \frac{r^{n-1}}{1 + r^{2n-1}} && \qquad (3.8)\\
&= (\alpha - \beta)\cdot\frac{1}{4}\cdot\frac{2kK}{\pi} && \text{from (3.6)}\\
&= \sqrt{p^2 - 4q}\cdot\frac{kK}{2\pi}.
\end{aligned}$$

Since the restrictions placed in w_{2n-1} in (3.7) are $A = B = 1$ and $q = -1$, formula (3.8) applies to sequences such as the odd-subscript Fibonacci (2.3) and Pell sequences. Accordingly,

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = \frac{\sqrt{5}kK}{2\pi} \quad \text{by (1.11)} \tag{3.9}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n-1}} = \frac{\sqrt{2}kK}{\pi} \quad \text{by (1.13).} \tag{3.10}$$

Because $r = \beta^2$ is different for $\{F_n\}$ and $\{P_n\}$, the term kK is different in (3.9) and (3.10).

Result (3.9) is not new and may be found in Catalan ([9], p. 13) while result (3.10), obtained by the author, appears in [26]. Bruckman ([7], p. 310) gave

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}} = 1.82451515\ldots \tag{3.9$'$}$$

while Bowen [4] obtained

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n-1}} = 1.24162540\ldots\ . \tag{3.10$'$}$$

Next, from (1.6) again

$$\begin{aligned} \frac{1}{w_{2n}} &= \frac{\alpha - \beta}{A(\alpha^{2n} - (B/A)\beta^{2n})} \\ &= \frac{\beta^{2n}}{(\alpha\beta)^{2n} + \beta^{4n}} && \text{if } A = -B = \alpha - \beta \text{ [cf. (2.9)]} \\ &= \frac{\beta^{2n}}{1 + \beta^{4n}} && \text{if } q = \pm 1 \text{ [cf. (1.5)]} \\ &= \frac{r^n}{1 + r^{2n}} && \text{where } \begin{cases} r = \beta^2 \ (\beta < 0 \text{ if } q = -1) \\ \sqrt{r} = |\beta|, \end{cases} \end{aligned} \tag{3.11}$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right) \quad \text{by (3.5).} \tag{3.12}$$

Under the constraints imposed on w_{2n} in (3.11), namely $A/B = -1$ and $q = \pm 1$, formula (3.12) applies to even-subscript Lucas (2.5) and Pell-Lucas sequences (with $q = -1$). Consequently,

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right) \tag{3.13}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = \frac{1}{4}\left(\frac{2K}{\pi} - 1\right), \tag{3.14}$$

the K being different in the two cases, since $r = \beta^2$ is different for $\{L_n\}$ and $\{Q_n\}$. However, notice that K in (3.9) [(3.10)] is the same as that in (3.13) [(3.14)]. Excluded from the summations are $1/L_0 = 1/Q_0 = 1/2$.

Result (3.13) occurs in Catalan ([9], p. 49) while (3.14) is given in [26]. Using essentially the same method, but checking results by a different method, Bruckman ([7], p. 310) has calculated

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n}} = 0.56617767\ldots \tag{3.13'}$$

and Bowen [4] found

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n}} = 0.20217495\ldots\ . \tag{3.14'}$$

Microcomputer calculations recorded above, and subsequently, which are due to my colleague, Dr. E. W. Bowen, are acknowledged with appreciation. All his computations were obtained using the recurrence relations for the sequences. Some of the numerical summations were found manually, to a lesser degree of accuracy, by the author.

Further standard information on Jacobian elliptic function theory may be found in Abramowitz and Stegun [1] and in Whittaker and Watson [29].

4. LAMBERT SERIES

The first reference to the series known as the Lambert series occurs in Lambert [22]—hence the name.

A "Lambert series" is a series of the type

$$\sum_{n=1}^{\infty} a_n \frac{x^n}{1 - x^n}. \tag{4.1}$$

Detailed information about Lambert series is to be found in Knopp [19] and [20]. Interesting number-theoretic applications (to primeness and divisibility), depending on the value of a_n, and some basic theory, are given in Knopp [20].

More particularly, we speak of *the Lambert series*

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} \qquad |x| < 1. \tag{4.2}$$

A *generalized Lambert series* used in Arista [2] is

$$L(a,\ x) = \sum_{n=1}^{\infty} \frac{ax^n}{1 - ax^n} \qquad |x| < 1,\ |ax| < 1, \tag{4.3}$$

where the number a has nothing to do with the initial value in (1.2). The series in (4.2) and (4.3) may be shown to be absolutely convergent within the indicated intervals of convergence.

From (1.6), we have

$$\frac{1}{w_{2n}} = \frac{\alpha - \beta}{A(\alpha^{2n} - (B/A)\beta^{2n})} \tag{4.4}$$

$$= (\alpha - \beta) \cdot \frac{\beta^{2n}}{(\alpha\beta)^{2n} - \beta^{4n}} \qquad \text{if } A = B = 1$$

$$= (\alpha - \beta) \cdot \frac{\beta^{2n}}{1 - \beta^{4n}} \qquad \text{if } q = \pm 1$$

$$= (\alpha - \beta)\left(\frac{\beta^{2n}}{1 - \beta^{2n}} - \frac{\beta^{4n}}{1 - \beta^{4n}}\right),$$

so

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n}} = (\alpha - \beta)\left\{\sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - \beta^{2n}} - \sum_{n=1}^{\infty} \frac{\beta^{4n}}{1 - \beta^{4n}}\right\} \tag{4.5}$$

$$= (\alpha - \beta)\{L(\beta^2) - L(\beta^4)\}.$$

To obtain (4.4) it was necessary to impose the conditions $A = B = 1$ and $q = \pm 1$. Accordingly, we can apply (4.5) to the even-subscript Fibonacci (2.4) and Pell sequences (where $q = -1$). It follows that

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{3 - \sqrt{5}}{2}\right) - L\left(\frac{7 - 3\sqrt{5}}{2}\right)\right] \tag{4.6}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n}} = 2\sqrt{2}[L(3 - 2\sqrt{2}) - L(17 - 12\sqrt{2})]. \tag{4.7}$$

Formula (4.6) has been known for a long time (cf. Catalan [9]), while (4.7) appears in [26].

It is known [4] that

$$\sum_{n=1}^{\infty} \frac{1}{P_{2n}} = 0.60057764\ldots . \tag{4.7$'$}$$

Brady [5] extended (4.6) to the summation $\sum_{n=1}^{\infty} (1/F_{2kn})$ and exhibited the graph of the function $y = L(x)$ for $|x| < 1$.

Let us now take a special case of $\{w_n\}$ which generalizes the Fibonacci sequence. Suppose in (1.1) we have $p = 1$, $q = -1$, and retain the initial values to be a and b. Call this sequence $\{H_n\}$, i.e., $H_0 = a$, $H_1 = b$. We impose the further condition: $b > a\alpha$, where $\alpha = (1 + \sqrt{5})/2$.

Write

$$H = \frac{A}{B} = \frac{b - a\beta}{b - a\alpha} \qquad \left(\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}\right). \tag{4.8}$$

Paralleling the argument in (4.4), we have

$$\frac{1}{H_{2n}} = \frac{(\alpha - \beta)\beta^{2n}}{A[(\alpha\beta)^{2n} - (B/A)\beta^{4n}]} = \frac{\sqrt{5}}{A} \cdot \frac{\beta^{2n}}{1 - (1/H)\beta^{4n}} \tag{4.9}$$

$$= \frac{\sqrt{5}}{A(1/\sqrt{H})} \cdot \frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/H)\beta^{4n}} = \frac{\sqrt{5}}{\sqrt{AB}}\left\{\frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/\sqrt{H})\beta^{2n}} - \frac{(1/H)\beta^{4n}}{1 - (1/H)\beta^{4n}}\right\}$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{H_{2n}} = \frac{\sqrt{5}}{\sqrt{AB}}\left\{\sum_{n=1}^{\infty} \frac{(1/\sqrt{H})\beta^{2n}}{1 - (1/\sqrt{H})\beta^{2n}} - \sum_{n=1}^{\infty} \frac{(1/H)\beta^{4n}}{1 - (1/H)\beta^{4n}}\right\} \tag{4.10}$$

$$= \frac{\sqrt{5}}{\sqrt{b^2 - ab - a^2}}\left\{L\left(\frac{1}{\sqrt{H}},\ \beta^2\right) - L\left(\frac{1}{H},\ \beta^4\right)\right\} \qquad \text{by (4.3)},$$

wherein $1/H_0$ has been omitted from the summation because a may be zero.

In (4.10), the conditions imposed in (4.3) are met, since

$$|\beta^2| < 1 \qquad \left(\beta = \frac{1-\sqrt{5}}{2} = -0.618\ldots\right)$$

and

$$\frac{1}{\sqrt{H}} = \sqrt{\frac{b - a\alpha}{b - a\beta}} < 1 \qquad (\alpha > 0,\ \beta < 0,\ b > a\alpha),$$

whence

$$\left|\frac{1}{\sqrt{H}}\beta^2\right| < 1; \text{ also, } |\beta^4| < 1,\ \left|\frac{1}{H}\beta^4\right| < 1.$$

Shannon and Horadam [28] obtained a variation of (4.10) by using a different pair of specially defined generalized Lambert series, whereas Arista's generalization (4.3) has been utilized in (4.10).

Observe that $\sqrt{AB}$ in (4.10) must be real, i.e., $AB > 0$. So (4.10) excludes Lucas-type sequences with $a = 2$, $b = 1$, 2, or 3, for which a Jacobian elliptic expression is required in the answer.

Suppose we introduce a generalized Pell sequence $\{K_n\}$ in which $p = 2$, $q = -1$, $b > a\alpha$, where $\alpha = 1 + \sqrt{2}$. Then, by reasoning similar to that used to establish (4.10), we can determine a resolution of $\sum_{n=1}^{\infty}(1/K_{2n})$ in terms of generalized Lambert series (4.3).

Let us now revert to the odd-subscript series contained in $\{L_n\}$ and $\{Q_n\}$. More generally, from (1.6)″, we have

$$\frac{1}{w_{2n-1}} = \frac{1}{\alpha^{2n-1} + \beta^{2n-1}} = \frac{\beta^{2n-1}}{(\alpha\beta)^{2n-1} + \beta^{4n-2}} \tag{4.11}$$

$$= -\frac{\beta^{2n-1}}{1 - \beta^{4n-2}} \qquad \text{for } q = -1 \text{ by (1.5)},$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{w_{2n-1}} = -\sum_{n=1}^{\infty} \frac{\beta^{2n-1}}{(1 - \beta^{4n-2})} = -L(\beta) + 2L(\beta^2) - L(\beta^4), \tag{4.12}$$

after some algebraic manipulation.

Thus, for appropriate β, expressions in terms of Lambert series as specializations of (4.12) are found for

$$\sum_{n=1}^{\infty} \frac{1}{L_{2n-1}} \qquad \left(\beta = \frac{1 - \sqrt{5}}{2}\right), \tag{4.13}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n-1}} \qquad (\beta = 1 - \sqrt{2}). \tag{4.14}$$

Bowen [4] calculated

$$\sum_{n=1}^{\infty} \frac{1}{Q_{2n-1}} = 0.58614901952408\ldots \tag{4.14$'$}$$

Furthermore, it was computed in [4] that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = 1.8422030498275\ldots \tag{4.15}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{Q_n} = 0.7883239758197\ldots\ . \tag{4.16}$$

Addition of (4.7)′ and (3.10)′ verifies (4.15), while addition of (3.14)′ and (4.14)′ leads us to (4.16).

To complete this section, we revert to an extension of $\{U_n\}$ (1.17) which Arista [2] examined in some depth. In his investigation, Arista imposed no restriction on q other than that it is a positive or negative integer. To avoid confusion with our notation, we will designate the sequence studied by Arista as $\{u_n\}$, where $u_0 = 0$, $u_1 = 1$, q being a positive or negative integer. Further, we will retain the condition $p^2 > 4q$, to avoid complex expressions, along with $p \geqslant 1$.

Changing to our notation, we record Arista's conclusions.

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = (\alpha - \beta) \sum_{h=0}^{\infty} \frac{\frac{\beta}{q}\left(\frac{\beta^2}{q}\right)^h}{1 - \frac{\beta}{q}\left(\frac{\beta^2}{q}\right)^h} = (\alpha - \beta)\left\{\frac{1}{\alpha - 1} + L\left(\frac{1}{\alpha}, \frac{\beta}{\alpha}\right)\right\}, \tag{4.17}$$

since $\left|\frac{\beta}{q}\right| < 1$, $\left|\frac{\beta}{q}\right|\left|\frac{\beta^2}{q}\right| < 1 \qquad [q = \alpha\beta \text{ (1.5)}]$.

If $q > 0$, then $\beta/\alpha > 0$, and Arista showed that (4.17) is then expressible in terms of a complicated definite integral involving logarithmic and trigonometrical functions.

When $q < 0$,

$$\sum_{n=1}^{\infty} \frac{1}{u_n} = (\alpha - \beta)\left\{\frac{1}{\alpha - 1} + L\left(\frac{1}{\alpha}, \left(\frac{\beta}{\alpha}\right)^2\right) + L\left(\frac{1}{\beta}, \left(\frac{\beta}{\alpha}\right)^2\right)\right\}, \tag{4.18}$$

which again leads to a lengthy expression containing indefinite integrals of the kind mentioned above.

Finally, the "degenerate" case in which the roots α, β are equal is considered as a limiting process to produce

$$\lim_{\alpha \to \beta} \sum_{n=1}^{\infty} \frac{1}{u_n} = \alpha \log\left(\frac{\alpha}{\alpha - 1}\right). \tag{4.19}$$

In the nondegenerate case ($\alpha \neq \beta$) Arista [2] also studied the consequences of $x \to 1$, and of $|a| < 1$. It is interesting to discern the usage made by him of the relevant researches of earlier and contemporary mathematicians, e.g., Cesàro [10], Schlömilch [27], and Catalan, *inter alia.*

Lucas [25] undertook to give *plus tard* (analogous) formulas deduced from the theory of elliptic functions, "*et, en particulier, les sommes des inverses des termes U_n et de leurs puissances semblables*". Writing a quarter of a century afterwards, Arista [2] remarked *à propos* this undertaking: "... *ma non esiste alcuna sua pubblicazione su questo argomento*".

5. APPLICATION OF METHODS OF GOOD AND GREIG

In this section we wish to develop some interesting techniques for summing reciprocals when the subscript of w (and of its specialized sequences) is not n, $2n$, or $2n - 1$, but is some related number.

Following an approach for Fibonacci numbers due to Good [12], we establish the corresponding result for Pell numbers:

$$\sum_{m=0}^{n} \frac{1}{P_{2^n}} = 2 - P_{2^n - 1}/P_{2^n}. \tag{5.1}$$

Proof of (5.1): The proof is by induction.

When $n = 1$, the result is obviously true, since

$$\frac{1}{P_1} + \frac{1}{P_2} \left(= 1 + \frac{1}{2}\right) = 2 - \frac{P_1}{P_2} \left(= 2 - \frac{1}{2}\right).$$

Assume it is true for $n = k$. Then the validity of (5.1) for $n = k + 1$ requires that

$$P_{2^k - 1}/P_{2^k} - P_{2^{k+1} - 1}/P_{2^{k+1}} = \frac{1}{P_{2^{k+1}}}.$$

This is readily demonstrated by using the Binet form for P_n [cf. (1.6)′ and (1.13)]. Thus, (5.1) is proved.

Now let $n \to \infty$. If, temporarily, $N = 2^n$, then $\lim_{n\to\infty}(P_{N-1}/P_N) = 1/\alpha = \sqrt{2} - 1$. Hence, (5.1) yields

$$\sum_{m=0}^{\infty} \frac{1}{P_{2^m}} = 3 - \sqrt{2}. \tag{5.2}$$

This might be compared with the corresponding value for Fibonacci numbers (Good [12]—see also Gould's reference [13], p. 67, to Millin):

$$\sum_{m=0}^{\infty} \frac{1}{F_{2^m}} = \frac{7 - \sqrt{5}}{2}. \tag{5.3}$$

Next, following the method and notation of Greig [14] for Fibonacci numbers, adapted for Pell numbers, let us write $b = 2^m$, $B = 2^n$. Then we may show that

$$\sum_{m=0}^{n} \frac{1}{P_{kb}} = C_k - P_{kB-1}/P_{kB} \qquad (n, k \geqslant 1), \tag{5.4}$$

where

$$C_k = \begin{cases} (1 + P_{k-1})/P_k & \text{for } k \text{ even}, \\ (1 + P_{k-1})/P_k + 2/P_{2k} & \text{for } k \text{ odd}, \end{cases} \tag{5.5}$$

i.e., C_k is independent of n.

Proof of (5.4): Again, the proof is by induction.

Assume (5.4) holds for a given n. Then its validity for $n + 1$ requires us to show that

$$P_{2kB}P_{kB-1} - P_{kB}P_{2kB-1} = P_{kB} \tag{5.6}$$

or, more succinctly, on writing $j = kB$,

$$P_{2j}P_{j-1} - P_jP_{2j-1} = (-1)^j P_j. \tag{5.6$'$}$$

This may be demonstrated by appealing to the Binet form for P_n.

[Alternatively, we may use

$$P_{h+1}P_j + P_hP_{j-1} = P_{h+j} \qquad (h = -2j,\ P_{-n} = (-1)^{n+1}P_n).] \tag{5.6$''$}$$

Put $n = 1$ in (5.4). Then

$$C_k = \frac{1}{P_k} + \frac{1 + P_{2k-1}}{P_{2k}} \tag{5.7}$$

$$= \begin{cases} (1 + P_{k-1})/P_k & \text{when } k \text{ is even}, \\ (1 + P_{k-1})/P_k + 2/P_{2k} & \text{when } k \text{ is odd}. \end{cases}$$

To obtain (5.7), we employ the Binet form in

$$\frac{1}{P_{2k}} + \frac{P_{2k-1}}{P_{2k}} - \frac{P_{k-1}}{P_k} = \begin{cases} 0 & \text{if } k \text{ is even}, \\ \dfrac{2}{P_{2k}} & \text{if } k \text{ is odd}. \end{cases} \tag{5.8}$$

Our proof of (5.4) is now complete.

The first few values of C_k are calculated from (5.7):

$$C_1 = 2,\ C_2 = 1,\ C_3 = \frac{22}{35},\ C_4 = \frac{1}{2},\ C_5 = \frac{534}{1189},\ \ldots . \tag{5.9}$$

Let $n \to \infty$. Then (5.4) becomes

$$\sum_{m=0}^{\infty} \frac{1}{P_{k \cdot 2^m}} = C_k - \frac{1}{\alpha}, \tag{5.10}$$

since $\lim_{n\to\infty}\left(\frac{P_{j-1}}{P_j}\right) = \frac{1}{\alpha} \qquad (j = kB = k \cdot 2^n;\ \alpha = 1 + \sqrt{2}).$

Observing from Gould [13] and Greig [14] that for $k \geqslant 0$, $m \geqslant 0$, $(2k+1)2^m$ generates each positive integer just once, we have (cf. [14]) that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{m=0}^{\infty} \frac{1}{P_{kb}} = \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \left(C_k - \frac{1}{\alpha}\right) \quad \left(\frac{1}{\alpha} = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1\right). \tag{5.11}$$

Summing the right-hand side of (5.11) as far as $k = 15$ (at which stage $C_{15} - 1/\alpha = 0.000005\ldots$), we find the value to six decimal places to be 1.842202... which concurs with the summation of $\sum_{n=1}^{20}(1/P_n)$. From these computations, we can state that

$$\sum_{n=1}^{\infty} \frac{1}{P_n} = 1.842202\ldots \tag{5.12}$$

approximately to six decimal places. See (4.15) for a slightly more accurate value.

One may observe that $C_k \to 1/\alpha$ as $k \to \infty$ on using the Binet form in (5.7), whence it follows that $C_{k+2}/C_k \to 1/\alpha^2$ as $k \to \infty$. This gives us an estimate for C_{k+2} when C_k is known, which increases in accuracy as k increases in value.

If one tries to parallel the above work for $\{Q_n\}$, one finds that the presence of the plus sign (rather than a minus sign) in the Binet form [cf. (1.6)″ and (1.14)] causes the straightforwardness of the treatment, e.g., at the stage (5.6), to collapse. A similar remark in relation to $\{L_n\}$ is made by Gould in [13], p. 68 (wherein the relation to the Riemann zeta function and to sine and cosine expressions is discussed).

Nevertheless, if we simply take a summation of reciprocals as far as $n = 20$, we obtain $\sum_{n=1}^{\infty}(1/Q_n)$ correct to six decimal places, namely, 0.7883239, as in (4.16).

Generalizing the results produced above for the Fibonacci-type sequences $\{F_n\}$ and $\{P_n\}$ to results for $\{w_n\}$ can be accomplished without too much effort.

Induction (details of which are available on request) can be applied to generate the following chain of formulas:

$$\sum_{m=0}^{n} \frac{1}{w_{2^m}} = C - w_{2^n - 1}/w_{2^n} \tag{5.13}$$

in which

$$C = \frac{1}{w_1} + \frac{1 + w_1}{w_2}; \tag{5.14}$$

$$\sum_{m=0}^{n} \frac{1}{w_{k \cdot 2^m}} = C_k - P_{k \cdot 2^n - 1}/P_{k \cdot 2^n} \qquad (n,\ k \geqslant 1) \tag{5.15}$$

where

$$C_k = \frac{1}{w_k} + \frac{1 + w_{2k-1}}{w_{2k}} = \begin{cases} (1 + w_{k-1})/w_k & \text{when } k \text{ is even,} \\ (1 + w_{k-1})/w_k + 2/w_{2k} & \text{when } k \text{ is odd;} \end{cases} \tag{5.16}$$

and

$$\sum_{m=0}^{\infty} \frac{1}{w_{k \cdot 2^m}} = C_k - \frac{1}{\alpha}, \tag{5.17}$$

where α is given by (1.4) ($q = -1$).

Note that, in (5.14),

$C = 3$ for Fibonacci numbers, $C = 2$ for Pell numbers.

For a generalization of (5.14) and (5.11), the reader might consult Greig [15]. Entries in row 2 of his table ([15], p. 257) give ratios of Pell numbers which are our C_1, C_2, C_3, ... in (5.9).

6. GENERALIZED BERNOULLI AND EULER POLYNOMIALS

In this final section, it is desired to find a suitable form for the expression of w_n^{-t} and for the generating function of $\{w_n^{-t}\}$. The results generalize material in [26] which itself extends the work in [28].

First, we define the *generalized Bernoulli polynomial* $B_r^{(t)}(x)$ by

$$\sum_{r=0}^{\infty} B_r^{(t)}(x)\frac{m^r}{r!} = \frac{m^t e^{mx}}{(e^m - 1)^t} \tag{6.1}$$

and the *generalized Euler polynomial* $E_r^{(t)}(x)$ by

$$\sum_{r=0}^{\infty} E_r^{(t)}(x)\frac{n^r}{r!} = \frac{2^t e^{nx}}{(e^n + 1)^t}. \tag{6.2}$$

When $t = 1$, $B_r^{(1)}(x) = B_r(x)$ and $E_r^{(1)}(x) = E_r(x)$ are the ordinary Bernoulli polynomial and Euler polynomial, respectively. Let

$$C = \frac{\beta}{\alpha}. \tag{6.3}$$

Temporarily write

$$m = n \log C \qquad (\text{i.e., } C^n = e^m). \tag{6.4}$$

From (1.6)′, for Fibonacci-type sequences,

$$\frac{1}{w_n^t} = (\beta - \alpha)^t \cdot \frac{1}{\alpha^{nt}(C^n - 1)^t} \tag{6.5}$$

$$= \frac{(\beta - \alpha)^t \cdot C^{nx}}{(C^x \alpha^t)^n (C^n - 1)^t} \qquad \text{introducing the variable } x$$

$$= \frac{(\beta - \alpha)^t}{m^t (C^x \alpha^t)^n} \cdot \frac{m^t e^{mx}}{(e^m - 1)^t} \qquad \text{by (6.4)}$$

$$= \frac{(\beta - \alpha)^t}{m\ (C^x \alpha\ )} \sum_{r=0}^{\infty} B_r^{(t)}(x) \frac{m^r}{r!} \qquad \text{by (6.1),}$$

whence arises the generating function

$$\sum_{n=1}^{\infty} \frac{1}{w_n^t} y^n = (\beta - \alpha)^t \sum_{r=0}^{\infty} B_r^{(t)}(x) \left(\frac{\log C}{r!}\right)^{r-t} \sum_{n=1}^{\infty} n^{r-t} \left(\frac{y}{\alpha^{t-x}\beta^x}\right)^n. \tag{6.6}$$

Putting $t = 1$ in (6.5) gives

$$\frac{1}{w_n} = \frac{(\beta - \alpha)}{(\alpha^{1-x}\beta^x)^n} \sum_{r=0}^{\infty} B_r(x) \frac{\left(\log\left(\frac{\beta}{\alpha}\right)\right)^{r-1}}{r!} n^{r-1}. \tag{6.7}$$

This expresses the reciprocal of appropriate w_n in terms of the Bernoulli polynomial.

A chain of results similar to (6.5)-(6.7) may be obtained from (1.6) and (6.2) for Lucas-type sequences. We then obtain an expression for the reciprocal of appropriate w_n in terms of the Euler polynomial.

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