DE MOIVRE-TYPE IDENTITIES FOR THE TRIBONACCI NUMBERS

PIN-YEN LIN
Taiwan Power Company
16F, 242 Roosevelt Road Section 3, Taipei 10763, R.O.C.

(Submitted February 1986)

1. INTRODUCTION

Recently *The Fibonacci Quarterly* has published a number of articles establishing for the Tribonacci sequence some analogs of properties of the Fibonacci sequence.

It is well known that, for $x^2 - x - 1 = 0$, the two roots are $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, and that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^n = \frac{L_n \pm \sqrt{5}F_n}{2} \tag{1}$$

as well as

$$\left(\frac{L_n \pm \sqrt{5}F_n}{2}\right)^m = \frac{L_{mn} \pm \sqrt{5}F_{mn}}{2}, \tag{2}$$

where L_n are the Lucas numbers and F_n are the Fibonacci numbers with m and n integers. Identities (1) and (2) are called "de Moivre-type" identities [9]. The purpose of this article is to establish de Moivre-type identities for the Tribonacci numbers.

2. DE MOIVRE-TYPE IDENTITIES FOR THE TRIBONACCI NUMBERS

From references [1] and [2], we get the three roots of $x^3 - x^2 - x - 1 = 0$. They are

$$r_1 = \frac{1}{3}(1 + X + Y), \tag{3}$$

$$r_2 = \frac{1}{3}\left[1 - \frac{3}{6}(X + Y) + \frac{3\sqrt{3}}{6}i\,(X - Y)\right], \tag{4}$$

and

$$r_3 = \frac{1}{3}\left[1 - \frac{3}{6}(X + Y) - \frac{3\sqrt{3}}{6}i\,(X - Y)\right], \tag{5}$$

where $X = \sqrt[3]{19 + 3\sqrt{33}}$ and $Y = \sqrt[3]{19 - 3\sqrt{33}}$. Using $X \cdot Y = 4$, and $X^3 + Y^3 = 38$, we have

$$r_1^2 = \frac{1}{3}\left[3 + \frac{2}{3}(X + Y) + \frac{1}{3}(X^2 + Y^2)\right],$$

$$r_1^3 = \frac{1}{3}\left[7 + \frac{5}{3}(X + Y) + \frac{1}{3}(X^2 + Y^2)\right],$$

$$r_1^4 = \frac{1}{3}\left[11 + \frac{10}{3}(X + Y) + \frac{2}{3}(X^2 + Y^2)\right],$$

$$r_1^5 = \frac{1}{3}\left[21 + \frac{17}{3}(X + Y) + \frac{4}{3}(X^2 + Y^2)\right],$$

and

$$r_1^6 = \frac{1}{3}\left[39 + \frac{32}{3}(X + Y) + \frac{7}{3}(X^2 + Y^2)\right].$$

The coefficients of the above equations are three Tribonacci sequences, which we denote by R_n, S_n, and T_n, respectively. The first ten numbers of these sequences are shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
R_n	3	1	3	7	11	21	39	71	131	241	443
S_n	3	2	5	10	17	32	59	108	199	366	673
T_n	1	1	2	4	7	13	24	44	81	149	274
U_n	0	1	2	3	6	11	20	37	68	125	230

By induction we establish that

$$r_1^n = \frac{1}{3}\left[R_n + \frac{S_{n-1}}{3}(X + Y) + \frac{T_{n-2}}{3}(X^2 + Y^2)\right]. \tag{6}$$

Using the same method, we obtain

$$r_2^n = \frac{1}{3}\left\{R_n - \frac{1}{6}[S_{n-1}(X + Y) + T_{n-2}(X^2 + Y^2)] + \frac{\sqrt{3}}{6}i\,[S_{n-1}(X - Y) + T_{n-2}(X^2 - Y^2)]\right\} \tag{7}$$

and

$$r_3^n = \frac{1}{3}\left\{R_n - \frac{1}{6}[S_{n-1}(X + Y) + T_{n-2}(X^2 + Y^2)] - \frac{\sqrt{3}}{6}i\,[S_{n-1}(X - Y) + T_{n-2}(X^2 - Y^2)]\right\}. \tag{8}$$

Hence, we find that r_1^n, r_2^n, and r_3^n can be expressed in terms of R_n, S_{n-1}, and T_{n-2}, so we have formulas equivalent to (1) for the Tribonacci numbers.

3. BINET'S FORMULA FOR R_n, S_n, AND T_n

From Spickerman [2] and Köhler [3], we can obtain Binet's formula for R_n, S_n, and T_n. That is,

$$R_n = r_1^n + r_2^n + r_3^n \tag{9}$$

and

$$S_n = d_1 r_1^n + d_2 r_2^n + d_3 r_3^n, \tag{10}$$

where $S_0 = 3$, $S_1 = 2$, and $S_2 = 5$.

From (10), it follows that

$$d_1 = \frac{3r_2 r_3 + 2r_1 + 3}{(r_1 - r_2)(r_1 - r_3)} = \frac{r_1(3r_1 - 1)}{(r_1 - r_2)(r_1 - r_3)},$$

$$d_2 = \frac{3r_3r_1 + 2r_2 + 3}{(r_2 - r_3)(r_2 - r_1)} = \frac{r_2(3r_2 - 1)}{(r_2 - r_3)(r_2 - r_1)},$$

$$d_3 = \frac{3r_1r_2 + 2r_3 + 3}{(r_3 - r_1)(r_3 - r_2)} = \frac{r_3(3r_3 - 1)}{(r_3 - r_1)(r_3 - r_2)},$$

and

$$T_n = \frac{r_1^{n+2}}{(r_1 - r_2)(r_1 - r_3)} + \frac{r_2^{n+2}}{(r_2 - r_3)(r_2 - r_1)} + \frac{r_3^{n+2}}{(r_3 - r_1)(r_3 - r_2)}. \tag{11}$$

T_n and R_n were originally discussed by Mark Feinberg [1] and Günter Köhler [3]. Equation (11) was derived by Spickerman [2].

4. SOME PROPERTIES OF R_n, S_n, AND T_n

As Ian Bruce shows in [6], using the Tribonacci sequence definition, some interesting results can be derived. We have also found the following:

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} \tag{12}$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3} \tag{13}$$

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \tag{14}$$

$$U_n = U_{n-1} + U_{n-2} + U_{n-3} \tag{15}$$

$$U_n = T_{n-1} + T_{n-2} \tag{16}$$

$$R_n = T_{n-1} + 2T_{n-2} + 3T_{n-3} \tag{17}$$

$$S_n = 3T_n - T_{n-1} \tag{18}$$

$$\sum_{i=1}^{n} U_i = T_{n+1} - 1 \tag{19}$$

$$\sum_{i=1}^{n} R_i = 2U_{n+2} + U_n - 3 \tag{20}$$

$$\sum_{i=1}^{n} S_i = \frac{3U_{n+1} + 2U_n - U_{n-1} - 2}{2} \tag{21}$$

$$\sum_{i=0}^{n} T_i = \frac{U_{n+2} + U_{n+1} - 1}{2} \tag{22}$$

$$T_0T_1 + T_1T_2 + T_2T_3 + T_3T_4 + \cdots + T_{n-1}T_n = \frac{U_n^2 + U_{n-1}^2 - 1}{4} \tag{23}$$

and

$$U_{4n+1}U_{4n+3} + U_{4n+2}U_{4n+4} = T_{4n+3}^2 - T_{4n+1}^2 \tag{24}$$

$$U_{n+1}^2 + U_{n-1}^2 = 2(T_{n-1}^2 + T_n^2) \tag{25}$$

$$T_n^2 - T_{n-1}^2 = U_{n+1} \cdot U_{n-1}. \tag{26}$$

ACKNOWLEDGMENT

The author is extremely grateful to the referee and to Mr. Hwang Kae Shyuan for their helpful comments and suggestions.

REFERENCES

1. M. Feinberg. "Fibonacci-Tribonacci." *The Fibonacci Quarterly 1*, no. 1 (1963):70-74.
2. W. R. Spickerman. "Binet's Formula for the Tribonacci Sequence." *The Fibonacci Quarterly 20*, no. 2 (1982):118-120.
3. G. Köhler. "Generating Functions of Fibonacci-Like Sequences and Decimal Expansions of Some Fractions." *The Fibonacci Quarterly 23*, no. 1 (1985): 29-35.
4. W. R. Spickerman & R. N. Joyner. "Binet's Formula for the Recursive Sequence of Order *K*." *The Fibonacci Quarterly 21*, no. 4 (1984):327-331.
5. C. P. McCarty. "A Formula for Tribonacci Numbers." *The Fibonacci Quarterly 19*, no. 5 (1981):391-393.
6. I. Bruce. "A Modified Tribonacci Sequence." *The Fibonacci Quarterly 22*, no. 3 (1984):244-246.
7. W. Gerdes. "Generalized Tribonacci Numbers and Their Convergent Sequences." *The Fibonacci Quarterly 16*, no. 3 (1978):269-275.
8. P. Lin. "The General Solution to the Decimal Fraction of Fibonacci Series." *The Fibonacci Quarterly 22*, no. 3 (1984):229-234.
9. M. Bicknell & V. E. Hoggatt, Jr., eds. *A Primer for the Fibonacci Numbers*. Santa Clara, Calif.: The Fibonacci Association, 1972, p. 45, B-10.

◆◇◆◇◆