SOME IDENTITIES FOR TRIBONACCI SEQUENCES

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1. INTRODUCTION

The sequence $\{F_n\}$ of Fibonacci numbers is defined by

$$F_0 = 0,\ F_1 = 1,$$

with the recurrence relation

$$F_{n+2} = F_{n+1} + F_n.$$

A number of identities for $\{F_n\}$ are well known. Among them are

$$F_{N-1}F_{N+1} - F_N^2 = (-1)^N \quad \text{and} \quad F_{N-1}F_{N+1} - F_{N-2}F_{N+2} = 2(-1)^N.$$

These identities were generalized by Harman in [1] by introducing the complex Fibonacci numbers. Similar generalized identities involving the combinations of the Fibonacci, Lucas, Pell, and Chebyshev sequences were obtained by this author (see [2]) by introducing the Generalized Gaussian Fibonacci Numbers defined using Harman's technique.

This gave rise to a natural question: Is it possible to achieve similar results for the Tribonacci numbers? This paper gives the answer in the affirmative. To achieve this, we define in Section 3 the complex Tribonacci numbers at the Gaussian integers. Our main result is equation (5.1).

2. TRIBONACCI NUMBER SEQUENCES

Denote by $\{S_n\}$ a sequence defined by the third-order recurrence relation given by

$$S_{n+3} = PS_{n+2} + QS_{n+1} + RS_n.$$

We consider the following particular cases of $\{S_n\}$ and call them the fundamental sequences of third order.

a. $\{J_n\}$ where $J_0 = 0$, $J_1 = 1$, and $J_2 = P$,

b. $\{K_n\}$ where $K_0 = 1$, $K_1 = 0$, and $K_2 = Q$,

c. $\{L_n\}$ where $L_0 = 0$, $L_1 = 0$, and $L_2 = R$.

If $P = Q = R = 1$, then $\{J_n\}$, $\{K_n\}$, and $\{L_n\}$ will be called the special fundamental sequences and will be denoted by $\{J_n^*\}$, $\{K_n^*\}$, and $\{L_n^*\}$, respectively.

The following relations are easily proved:

$$H_{n+1} = PJ_n + K, \quad n \geqslant 0; \tag{2.1}$$

$$K_{n+1} = QJ_n + RJ_{n-1}, \quad n \geqslant 1; \tag{2.2}$$

$$L_{n+1} = RJ_n, \quad n \geqslant 0. \tag{2.3}$$

By (2.3), (2.2) can also be written as

$$K_{n+1} = QJ_n + L_n. \tag{2.4}$$

It is helpful to know the first few terms of the above sequences. We present them in Table 2.1. These sequences have been studied by many researchers (see, e.g., Shannon [3], Shannon & Horadam [4], and Waddill & Sacks [5]).

Table 2.1

$\{S_n\}$ \ n	0	1	2	3	4	5	6
$\{J_n\}$	0	1	P	P^2+Q	$P^3+2PQ+R$	$P^4+3P^2Q+2PR+Q^2$	$P^5+4P^3Q+3P^2R+3PQ^2+2QR$
$\{K_n\}$	1	0	Q	$PQ+R$	$P^2Q+PR+Q^2$	$P^3Q+P^2R+2PQ^2+2QR$	$P^4Q+P^3R+3P^2Q^2+4PQR+Q^3+R^2$
$\{L_n\}$	0	0	R	PR	P^2R+QR	$P^3R+2PQR+R^2$	$P^4R+3P^2QR+2PR^2+Q^2R$

3. DEFINITION

Let (n, m), $n, m \in \mathbb{Z}$, denote the set of Gaussian integers $(n, m) = n + im$. Let $G: (n, m) \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, be a function defined as follows:

For fixed real numbers P, Q, and R, define

$$\begin{cases} G(0, 0) = 0, \; G(1, 0) = 1, \; G(2, 0) = P \\ G(0, 1) = i, \; G(1, 1) = P + iP, \; G(2, 1) = P^2 + i(P^2 + Q) \\ G(0, 2) = iP, \; G(1, 2) = P^2 + Q + iP^2, \; G(2, 2) = P^3 + PQ + i(P^3 + PQ) \end{cases} \tag{3.1}$$

with the following conditions:

$$G(n + 3, m) = PG(n + 2, m) + QG(n + 1, m) + RG(n, m), \tag{3.2}$$

and

$$G(n, m + 3) = PG(n, m + 2) + QG(n, m + 1) + RG(n, m). \tag{3.3}$$

The conditions (3.2) and (3.3) with the initial values (3.1) are sufficient to obtain a unique value for every Gaussian integer with nonnegative values for n and m.

4. RESULTS CONCERNING $G(n, m)$

Lemma 4.1: $G(n, 0)$ and $G(0, m)$ are given by

$$G(n, 0) = J_n, \text{ and } G(0, m) = iJ_m. \tag{4.1}$$

Proof: The proof is simple and hence omitted.

Theorem 4.2: $G(n, m)$ is given by

$$G(n, m) = J_n J_{m+1} + iJ_{n+1}J_m. \tag{4.2}$$

Proof: Although an elegant proof can be given by using the technique of mathematical induction, we give another below, which although not so elegant brings out more clearly the interaction. We have:

$$\begin{aligned} G(n, m) &= PG(n-1, m) + QG(n-2, m) + RG(n-3, m) \\ &= P\{PG(n-2, m) + QG(n-3, m) + RG(n-4, m)\} \\ &\qquad + QG(n-2, m) + RG(n-3, m) \\ &= (P^2 + Q)G(n-2, m) + (PQ + R)G(n-3, m) + PRG(n-4, m) \\ &= J_3G(n-2, m) + K_3G(n-3, m) + L_3G(n-4, m) \\ &= J_3[PG(n-3, m) + QG(n-4, m) + RG(n-5, m)] \\ &\qquad + K_3G(n-3, m) + L_3G(n-4, m) \\ &= (PJ_3 + K_3)G(n-3, m) + (QJ_3 + L_3)G(n-4, m) + RJ_3G(n-5, m) \end{aligned}$$

Now we make use of (2.1), (2.4), and (2.3) to set

$$G(n, m) = J_4G(n-3, m) + K_4G(n-4, m) + L_4G(n-5, m).$$

Continuing this process, we finally get

$$G(n, m) = J_{n-1}G(2, m) + K_{n-1}G(1, m) + L_{n-1}G(0, m). \tag{4.3}$$

We apply the same technique for $G(2, m)$, $G(1, m)$, and $G(0, m)$ to get

$$G(2, m) = J_{m-1}G(2, 2) + K_{m-1}G(2, 1) + L_{m-1}G(2, 0),$$

$$G(1, m) = J_{m-1}G(1, 2) + K_{m-1}G(1, 1) + L_{m-1}G(1, 0),$$

and $$G(0, m) = J_{m-1}G(0, 2) + K_{m-1}G(0, 1) + L_{m-1}G(0, 0).$$

Then (3.1) gives

$$\begin{cases} G(2, m) = \{P^3 + PQ + i(P^3 + PQ)\}J_{m-1} + [P^2 + i(P^2 + Q)]K_{m-1} + PL_{m-1}, \\ G(1, m) = (P^2 + Q + iP^2)J_{m-1} + (P + iP)K_{m-1} + L_{m-1}, \text{ and} \\ G(0, m) = iPJ_{m-1} + iK_{m-1}. \end{cases} \tag{4.4}$$

Substituting the values of $G(2, m)$, $G(1, m)$, and $G(0, m)$ from (4.4) into (4.3) and simplifying, we get:

$$\begin{aligned} G(n, m) &= \{(P^3 + PQ)J_{n-1} + (P^2 + Q)K_{n-1} + i[(P^3 + PQ)J_{n-1} \\ &\qquad + P^2K_{n-1} + PL_{n-1}]\}J_{m-1} \\ &\quad + \{P^2J_{n-1} + PK_{n-1} + i[(P^2 + Q)J_{n-1} + PK_{n-1} + L_{n-1}]\}K_{m-1} \\ &\quad + \{PJ_{n-1} + K_{n-1}\}L_{m-1} \end{aligned}$$

Using equations (2.1)-(2.4), we obtain:

$$\begin{aligned} G(n, m) &= J_{m-1}\{P^2J_n + QJ_n + i(P^2J_n + PK_n)\} \\ &\qquad + K_{m-1}\{PJ_n + i[PJ_n + K_n]\} + L_{m-1}J_n \\ &= J_n\{P^2J_{m-1} + QJ_{m-1} + PK_{m-1} + L_{m-1} + i[P^2J_{m-1} + PK_{m-1}]\} \\ &\qquad + iK_n\{PJ_{m-1} + K_{m-1}\} \\ &= J_nJ_{m+1} + iJ_{n+1}J_m \end{aligned}$$

Theorem 4.3: For fixed n, m (n, m = 0, 1, ...), the recurrence relation for $G(n, m)$ is given by the following:

$$\begin{aligned} G(n+k, m+k) = {} & (P + iP)\sum_{j=1}^{k} Q^{k-j} J_{n+j} J_{m+j} \\ & + R\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-s} J_{m+2j-2-s} \right. \\ & \left. + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-3+s} J_{m+2j-1+s}\right] \\ & + iR\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-2-s} J_{m+2j-s} \right. \\ & \left. + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-1+s} J_{m+2j-3+s}\right] \\ & + Q^k \begin{cases} G(n, m), & \text{if } k \text{ is even} \\ G(m, n), & \text{if } k \text{ is odd,} \end{cases} \end{aligned} \tag{4.5}$$

where $s = \begin{cases} 0, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd} \end{cases}$ and $[k/2]$ denotes the greatest integer function.

Proof: Fix n and m. From (4.2), we have:

$$\begin{aligned} G(n+1, m+1) &= J_{n+1}J_{m+2} + iJ_{n+2}J_{m+1} \\ &= J_{n+1}[PJ_{m+1} + QJ_m + RJ_{m-1}] + i[PJ_{n+1} + QJ_n + RJ_{n-1}]J_{m+1} \end{aligned}$$

By algebraic manipulation and interchanging n and m in (4.2), we get

$$\begin{aligned} G(n+1, m+1) = {} & (P + iP)J_{n+1}J_{m+1} + RJ_{n+1}J_{m-1} \\ & + iRJ_{n-1}J_{m+1} + QG(m, n). \end{aligned} \tag{4.6}$$

Similarly, we have

$$\begin{aligned} G(n+2, m+2) = {} & (P + iP)[J_{n+2}J_{m+2} + QJ_{n+1}J_{m+1}] \\ & + R[J_{n+2}J_m + QJ_{n-1}J_{m+1}] \\ & + iR[J_nJ_{m+2} + QJ_{n+1}J_{m-1}] + Q^2G(n, m). \end{aligned} \tag{4.7}$$

(4.6) and (4.7) show that (4.5) holds for $k = 1$ and $k = 2$. Now, suppose (4.5) holds for the first k positive integers. We prove that then it also holds for the integer $k + 2$. Now, although n and m are assumed to be fixed in (4.7), it is clear that (4.7), in fact, is true for any positive integers n and m. Thus, replacing n and m by $n + k$ and $m + k$, respectively, in (4.7), we get:

$$\begin{aligned} G(n+k+2, m+k+2) = {} & (P + iP)[J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1}] \\ & + R[J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1}] \\ & + iR[J_{n+k}J_{m+k+2} + QJ_{n+k+1}J_{m+k-1}] \\ & + Q^2G(n+k, m+k) \end{aligned}$$

Substituting for $G(n + k, m + k)$ from (4.5), we get:

$$\begin{aligned} G(n+k+2, m+k+2) = {} & (P + iP)[J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1}] \\ & + R[J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1}] \\ & + iR[J_{n+k}J_{m+k+2} + QJ_{n+k+1}J_{m+k-1}] \\ & + Q^2\Bigg\{(P + iP)\sum_{j=1}^{k} Q^{k-j}J_{n+j}J_{m+j} \\ & + R\Bigg[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s}J_{n+2j-s}J_{m+2j-2-s} \\ & + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s}J_{n+2j-3+s}J_{m+2j-1+s}\Bigg] \\ & + iR\Bigg[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s}J_{n+2j-2-s}J_{m+2j-s} \\ & + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s}J_{n+2j-1+s}J_{m+2j-3+s}\Bigg] \\ & + Q^k\begin{cases} G(n, m), & k \text{ even} \\ G(m, n), & k \text{ odd} \end{cases}\Bigg\} \end{aligned} \tag{4.8}$$

We observe the following on the right-hand side of (4.8):

The coefficient of $P + iP$ is

$$J_{n+k+2}J_{m+k+2} + QJ_{n+k+1}J_{m+k+1} + \sum_{j=1}^{k} Q^{k+2-j}J_{n+j}J_{m+j}$$
$$= \sum_{j=1}^{k+2} Q^{k+2-j}J_{n+j}J_{m+j}.$$

The coefficient of R is

$$J_{n+k+2}J_{m+k} + QJ_{n+k-1}J_{m+k+1} + \sum_{j=1}^{[k/2]+s} Q^{k+2-2j+s}J_{n+2j-s}J_{m+2j-2-s}$$
$$+ \sum_{j=1}^{[k/2]} Q^{k+3-2j-s}J_{n+2j-3+s}J_{m+2j-1+s}.$$

Observing that, if $j = [k/2] + 1 + s$ and $j = [k/2] + 1$, $2j = k + 2 + s$ and $k + 2 - s$, respectively, where s is as defined before, we see that:

The coefficient of R is

$$\sum_{j=1}^{[k/2]+1+s} Q^{k+2-2j+s}J_{n+2j-s}J_{m+2j-2-s}$$
$$+ \sum_{j=1}^{[k/2]+1} Q^{k+3-2j-s}J_{n+2j-3+s}J_{m+2j-1+s}.$$

Similarly:

The coefficient of iR is

$$\sum_{j=1}^{[k/2]+1+s} Q^{k+2-2j+s} J_{n+2j-2-s} J_{m+2j-s} + \sum_{j=1}^{[k/2]+1} Q^{k+3-2j-s} J_{n+2j-1+s} J_{m+2j-3+s}.$$

The last term is

$$Q^{k+2}\begin{cases} G(n, m), & k \text{ even}, \\ G(m, n), & k \text{ odd}. \end{cases}$$

These coefficients are exactly the same as those, respectively, on the right-hand side of (4.5) with k replaced by $k + 2$. This completes the proof.

5. IDENTITIES FOR $\{J_n\}$

Equating the real parts of (4.5), and making use of (4.2), we get:

$$P\sum_{j=1}^{k} Q^{k-j} J_{n+j} J_{m+j} + R\left[\sum_{j=1}^{[k/2]+s} Q^{k-2j+s} J_{n+2j-s} J_{m+2j-2-s} + \sum_{j=1}^{[k/2]} Q^{k-2j+1-s} J_{n+2j-3+s} J_{m+2j-1+s}\right] = J_{n+k} J_{m+k+1} - Q^k J_{n+s} J_{m+1-s} \tag{5.1}$$

Remark 1: Equation (5.1) gives the sum of $2k$ terms as that of just two terms. Note that equating the imaginary parts of (4.5) gives (5.1) with m and n interchanged and, therefore, effectively the same equation.

We now consider some special cases.

6. SPECIAL CASES

(A) $m = n$

Putting $s = 0$ and $s = 1$, in turn, for k even and k odd, respectively, we readily observe that, for both even and odd k, (5.1) reduces to a single equation given by

$$P\sum_{j=1}^{k} Q^{k-j} J_{n+j}^2 + R\sum_{j=1}^{k} Q^{k-j} J_{n+j-2} J_{n+j} = J_{n+k} J_{n+k+1} - Q^k J_n J_{n+1}. \tag{6.1}$$

(B) $m = n = 0$

With these values of m and n, (6.1) reduces to

$$P\sum_{j=1}^{k} Q^{k-j} J_j^2 + R\sum_{j=1}^{k} Q^{k-j} J_{j-2} J_j = J_k J_{k+1}. \tag{6.2}$$

(C) $n = 1$, $m = 0$

Equation (5.1) takes the following form:

$$P\sum_{j=1}^{k} Q^{k-j} J_j J_{j+1} + R\left\{\sum_{j=1}^{[k/2]} Q^{k-2j} J_{2j-2+2s}\,[QJ_{2j-1-2s} + J_{2j+1-2s}]\right\} \tag{6.3}$$

$$= \begin{cases} J_{k+1}^2 - Q^k & \text{if } k \text{ is even,} \\ J_{k+1}^2 - RJ_{k+1}J_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

Remark 2: Various other identities may be obtained for other choices of m and n. Thus, equation (5.1) provides an abundance of identities.

Remark 3: If $P = Q = R = 1$, the identities in Sections 5 and 6 reduce to those for the "special fundamental sequences." It is interesting to compare these identities with similar ones for Fibonacci sequences. For example, for $n = 0$, $m = 0$, (6.2) becomes

$$\sum_{j=1}^{k} J_j^{*2} + \sum_{j=1}^{k} J_{j-2}^{*} J_j^{*} = J_k^{*} J_{k+1}^{*},$$

and for $n = 1$, $m = 0$, (6.3) reduces to

$$\sum_{j=1}^{k} J_j^{*} J_{j+1}^{*} + \sum_{j=1}^{[k/2]} J_{2j-2+2s}^{*}\,[J_{2j-1-2s}^{*} + J_{2j+1-2s}^{*}]$$

$$= \begin{cases} J_{k+1}^{*2} - 1, & \text{if } k \text{ is even,} \\ J_{k+1}^{*2} - J_{k+1}^{*} J_{k-2}^{*}, & \text{if } k \text{ is odd.} \end{cases}$$

Similar identities for the Fibonacci sequence are

$$\sum_{j=1}^{k} F_j^2 = F_k F_{k+1},$$

and
$$\sum_{j=1}^{k} F_j F_{j+1} = \begin{cases} F_{k+1}^2 - 1, & \text{if } k \text{ is even,} \\ F_{k+1}^2, & \text{if } k \text{ is odd.} \end{cases}$$

(See [1].)

Remark 4: If $R = 0$, the sequence $\{J_n\}$ reduces to the sequence with second-order recurrence relation. If, in addition, $P = p$ and $Q = -q$, $\{J_n\}$ becomes Lucas's fundamental sequence [2]. If $P = 1$ and $Q = 1$, $\{J_n\}$ reduces to the Fibonacci sequence. In these cases, equation (5.1) and the rest of the equations reduce to equation (5.1) and the others, respectively, of [2].

Remark 5: Define the initial terms as follows:

$G(0, 0) = 0$, $G(1, 0) = iQ$, $G(2, 0) = i(PQ + R)$

$G(0, 1) = Q$, $G(1, 1) = 0$, $G(2, 1) = Q^2$

$G(0, 2) = PQ + R$, $G(1, 2) = iQ^2$, $G(2, 2) = Q(PQ + R) + iQ(PQ + R)$

Then, following a technique similar to that used in Theorem 4.2, we prove that

$$G(n, m) = K_n K_{m+1} + iK_{n+1}K_m. \tag{6.4}$$

Since (6.4) is exactly the same as (4.2) with J_i replaced by K_i, it can be readily seen that with such a replacement all identities proved in Sections 5 and 6 can be transformed into ones with $\{K_n\}$ and $\{K_n^*\}$. The same is true for $\{L_n\}$ and $\{L_n^*\}$.

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