ON A RESULT INVOLVING ITERATED EXPONENTIATION

R. M. STERNHEIMER
Brookhaven National Laboratory, Upton, NY 11973

(Submitted July 1986)

In connection with recent work by M. Creutz and myself involving iterated exponentiation [1], [2], [3], e.g., the function

$$f(x) = x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}, \tag{1}$$

with an infinite number of x's, I have noticed an interesting property when only a finite number n of x's is considered.

I will now consider the bracketing a for $n = 4$. This is defined as

$$F_{4,a}(x) \equiv x^{[x^{(x^x)}]} = {}^4x. \tag{2}$$

In a Brookhaven National Laboratory Report [4], I have given a more extensive discussion of the present results (see, in particular, Table 1 of [4]). Obviously, when $x > 2$, the function $F_{4,a}(x)$ has a large numerical value. As an example, we consider

$$F_{4,a}(5) = 5^{[5^{(5^5)}]} = 5^{(5^{3125})}. \tag{3}$$

Now we find

$$5^{3125} \simeq 10^{2184.281} = 1.910 \times 10^{2184}, \tag{4}$$

where

$$2184.281 = 5^5 \log_{10} 5 = (3125)(0.69897). \tag{5}$$

From equations (3)-(5), one obtains

$$F_{4,a}(5) = 5^{(10^{2184.281})} = 5^{1.910 \times 10^{2184}} \tag{6}$$

A seemingly paradoxical result is obtained if we express $F_{4,a}(5)$ as a power of 10. Thus, we find the exponent

$$\log_{10}[5^{(10^{2184.281})}] = 10^{2184.281} \log_{10} 5 = 0.69897 \times 1.910 \times 10^{2184}$$
$$\doteq 1.335 \times 10^{2184} = 10^{2184.125}, \tag{7}$$

This manuscript has been authored under contract number DE-AC02-76CH00016 with the U.S. Department of Energy.

which leads to the result

$$F_{4,a}(5) = 10^{(10^{2184.125})}, \tag{8}$$

showing [by comparison with (6)] that the exponent in the parentheses is hardly changed in going from a power of 5 to a power of 10.

To clarify this result, we consider the equation

$$x^{(10^y)} = 10^{(10^{y'})}, \tag{9}$$

which defines y', where in the present case $x = 5$ and $y = 2184.281$. To derive the relationship between y' and y, we take the logarithms of both sides of (9). This gives

$$10^y \log_{10} x = 10^{y'}, \tag{10}$$

By taking the logarithms of both sides of this equation, we obtain

$$y' = y + \log_{10}\log_{10} x. \tag{11}$$

For the case discussed above, it can be readily verified that $\log_{10}\log_{10} 5 = -0.1555$, leading to the results in (6) and (8), since $0.281 - 0.125 = 0.156$, which is clearly consistent with the value of $\log_{10}\log_{10} 5 = -0.1555$ obtained above. It is of interest that the correction to y, namely $\log_{10}\log_{10} x$, is independent of the value of y.

To make the above results more believable, note that the *ratio* of the two powers of 10 involved in (6) and (7) above is given by

$$R = 10^{2184.281} / 10^{2184.125} = 10^{0.156} = 1.432. \tag{12}$$

Thus, the very large exponent $10^{2184.125}$ is multiplied by 1.432 in going from $x = 10$ to $x = 5$. This is a very considerable increase. As a result, we write

$$5^{1.432 \times 10^{2184.125}} = 10^{10^{2184.125}}, \tag{13}$$

which is essentially correct because $5^{1.432} = 10.02$. (The small apparent discrepancy of 0.02 is due to rounding errors.)

As a final comment, I note that, if I had used $x = 1.1$ (instead of 5.0), with the correction $\log_{10}\log_{10} 1.1 = -1.383$, and $y' = 2184.125 + 1.383 = 2185.508$, I would have obtained

$$10^{10^{2184.125}} = 1.1^{10^{2185.508}}, \tag{14}$$

since $10^{1.383} = 24.15$ and $1.1^{24.15} \simeq 10$.

ACKNOWLEDGMENT

I would like to thank Dr. Michael Creutz for helpful discussions and for a critical reading of the manuscript.

REFERENCES

1. R.M. Sternheimer. "On a Set of Non-Associative Functions of a Single Positive Real Variable." Brookhaven Informal Rpt. PD-128; BNL-23081 (June 1977).
2. M. Creutz & R.M. Sternheimer. "On the Convergence of Iterated Exponentiation—I." *The Fibonacci Quarterly 18*, no. 4 (1980):341-347.
3. M. Creutz & R.M. Sternheimer. "On the Convergence of Iterated Exponentiation—II." *The Fibonacci Quarterly 19*, no. 4 (1981):326-335.
4. R.M. Sternheimer. "On a Set of Non-Associative Functions of a Single Positive Variable." Brookhaven National Laboratory Rpt. BNL-38485 (July 1986).

◆◇◆◇◆