FIBONACCI SEQUENCES OF SETS AND THEIR DUALS

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In this paper, Fibonacci sequences of sets and their duals are defined and used first to obtain short proofs of two well-known theorems on the representation of integers as sums of Fibonacci numbers, and second to produce two sets of binary numbers that resemble Cantor's ternary set. It is also shown how Fibonacci sequences of sets and their duals can be represented by sequences of trees.

Given any sequence $C = (c_1, c_2, \ldots)$ of real numbers, let the corresponding Fibonacci sequence of sets and its dual be defined by

$$S_0 = \{0\},\ S_1 = \{c_1\},\ S_n = \{x : (x - c_n) \in (S_{n-1} \cup S_{n-2})\}, \tag{1}$$

and

$$S'_0 = \{0\},\ S'_1 = \{c_1\},\ S'_n = \{x : (x - c_n) \in (S_0 \cup S_1 \cup \cdots \cup S_{n-2})\}. \tag{1'}$$

These definitions resemble the recurrence relations that may be used to define the sequence $F = (u_1, u_2, \ldots)$ of distinct positive Fibonacci numbers, namely,

$$u_0 = u_1 = 1,\quad u_n = u_{n-1} + u_{n-2}; \tag{2}$$

$$u_0 = u_1 = 1,\quad u_n = 1 + u_0 + u_1 + \cdots + u_{n-2}. \tag{2'}$$

The following lemmas are easily proved by induction.

Lemma 1: $x \in S_n$ if and only if x is of the form

$$x = \sum_{j=1}^{n} e_j c_j, \quad n \geqslant 1, \tag{3}$$

where

$$e_j \in \{0, 1\},\quad e_n = 1,\quad e_j + e_{j+1} \neq 0 \text{ if } 1 \leqslant j < n. \tag{4}$$

There are exactly u_n distinct n-tuples $(e_1, \ldots, e_n)$ satisfying (4).

Lemma 1′: $x \in S'$ if and only if x is of the form (3), where

$$e_j \in \{0, 1\},\quad e_n = 1,\quad e_j e_{j+1} = 0 \text{ if } 1 \leqslant j < n. \tag{4'}$$

There are exactly u_{n-1} distinct n-tuples $(e_1, \ldots, e_n)$ satisfying (4′) if $n \geqslant 1$.

Two special choices of C are of interest. The first choice, $C = F$, yields short proofs of two well-known theorems.

Theorem 1 (Brown [1]): Every positive integer has one and only one representation (the so-called Dual of the Zeckendorf representation) in the form

$$x = \sum_{j=1}^{n} e_j u_j, \quad n \geq 1, \tag{5}$$

where $(e_1, \ldots, e_n)$ satisfies (4).

Theorem 1′ (Lekkerkerker [2]): Every positive integer has one and only one representation (the so-called Zeckendorf representation) in the form (5), where $(e_1, \ldots, e_n)$ satisfies (4′).

Proofs: Let $C = F$ and let S_n and S_n' be defined by (1) and (1′). It is seen, by induction on n, that

$$S_n = \{u_{n+1} - 1, u_{n+1}, u_{n+1} + 1, \ldots, u_{n+2} - 2\},$$

and

$$S_n' = \{u_n, u_n + 1, u_n + 2, \ldots, u_{n+1} - 1\},$$

for $n = 1, 2, 3, \ldots$. Theorems 1 and 1′ now follow from Lemmas 1 and 1′.

The second choice of C is $C = B$, where

$$B = \left(\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\right). \tag{6}$$

We now show that this choice leads to two binary sets that resemble Cantor's ternary set.

Theorem 2: Let $\overline{S}$ be the set of all real numbers x whose binary expansion is $x = 0 \cdot e_1 e_2 \ldots$, where $e_j + e_{j+1} \neq 0$ for all $j \geq 1$ if the expansion does not terminate, for $1 \leq j < n$ if the expansion terminates with the digit $e_n = 1$. Then $\overline{S}$ is an uncountable closed set of measure 0.

Theorem 2′: Let $\overline{S'}$ be the set of all real numbers x whose binary expansion is $x = 0 \cdot e_1 e_2 \ldots$, where $e_j e_{j+1} = 0$ for all $j \geq 1$. Then $\overline{S'}$ is an uncountable closed set of measure 0.

Proofs: Let $C = B$, defined by (6). Let S_n and S_n' be defined by (1) and (1′). Let

$$S = \bigcup_{n=1}^{\infty} S_n, \quad S' = \bigcup_{n=1}^{\infty} S_n'.$$

By Lemma 1 (Lemma 1′), S_n (S_n') contains exactly the binary fractions in $\overline{S}$ $(\overline{S'})$ that terminate with the digit $e_n = 1$, and it is clear that $\overline{S}$ $(\overline{S'})$ is the closure of S (S'). Also, it is easily seen that

$$\overline{S} \subseteq \left[\frac{1}{4}, 1\right] \quad \text{and} \quad \overline{S'} \subseteq \left[0, \frac{2}{3}\right].$$

Now $z \in [1/4, 1] - \overline{S}$ if and only if z is a binary fraction of the form

$$z = 0 \cdot e_1 e_2 \ldots e_n 0\ 0 e_{n+3} \ldots,$$

where $0 \cdot e_1 e_2 \ldots e_n \in S_n$, $n \geqslant 1$, and $e_m = 1$ for at least one $m \geqslant n + 3$. It follows that the complement of S in $[1/4, 1]$ is the open set

$$C = \left[\frac{1}{4}, 1\right] - \overline{S} = \bigcup_{n=1}^{\infty} \bigcup_{x \in S_n} (x, x + 2^{-n-2}). \tag{7}$$

The intervals on the right of (7) are disjoint because their end-points belong to $\overline{S}$, and their total length is

$$\sum_{n=1}^{\infty} \frac{1}{2^{n+2}} u_n = \frac{3}{4}$$

by Lemma 1 and the well-known result

$$\sum_{n=0}^{\infty} u_n x^n = \frac{1}{1 - x - x^2} \text{ if } |x| < \frac{\sqrt{5} - 1}{2}.$$

It follows readily that $\overline{S} = [1/4, 1] - C$ has measure 0.

While S is clearly countable, $\overline{S}$ is not. For, if

$$0 \cdot e_{k,1} e_{k,2} \ldots \quad (k = 1, 2, \ldots)$$

is any countable list of elements of $\overline{S}$ in binary notation, consider

$$x = 0 \cdot e_1 e_2 \ldots,$$

where $(e_{3k-2}, e_{3k-1}, e_{3k}) = (1, 0, 1)$ or $(1, 1, 1)$ according as

$$(e_{k,3k-2}, e_{k,3k-1}, e_{k,3k}) = (1, 1, 1)$$

or not; clearly, x belongs to $\overline{S}$ but does not occur in the list.

Before proceeding to the proof of Theorem 2′, note that $\overline{S}$ can be written as the disjoint union

$$\overline{S} = \overline{S}^* \cup \overline{S}^{**},$$

where $\overline{S}^{**}$ is the set consisting of all elements of $\overline{S}$ whose binary expansion terminates with 01, and where $\overline{S}^* = \overline{S} - \overline{S}^{**}$. Clearly, $\overline{S}^{**}$ is countable, and it is easily seen that $\overline{S}^{**}$ consists of all the isolated points of $\overline{S}$, while $\overline{S}^*$ consists of all the limit points of $\overline{S}$. Like $\overline{S}$, $\overline{S}^*$ is, therefore, an uncountable, closed set of measure 0. Thus, Theorem 2′ follows from Theorem 2, since $x \in \overline{S'}$ if and only if $1 - x \in \overline{S}^*$. However, it is interesting to note that, if $x \in S_n'$, $n \geqslant 1$, then

$$\left(x - \frac{1}{3} 2^{-n}, x\right) \subseteq C', \text{ where } C' = \left[0, \frac{2}{3}\right] - \overline{S'}.$$

Conversely, suppose that $z \in C'$. Then z must be a binary fraction of the form

$$z = e_0 \cdot e_1 e_2 \ldots e_m e_{m+1} \ldots,$$

where $e_0 \cdot e_1 e_2 \ldots e_m \in S_m'$ and $e_{m+1} = 1$. Let n be the largest subscript such that $1 \leqslant n < m$, $e_{n-1} = e_n = 0$, and $e_{n+1} = 1$. This n exists because

$$e_0 \cdot e_1 e_2 \ldots e_m < z < \frac{2}{3}.$$

Put $x = e_0 \cdot e_1 e_2 \ldots e_{n-1} 1$. Then $x \in S_n'$, $n \geqslant 1$, and

$$z \in \left(x - \frac{1}{3}\, 2^{-n},\ x\right).$$

It follows that the complement of $\overline{S'}$ in $[0, 2/3]$ is the open set

$$C' = \left[0, \frac{2}{3}\right] - \overline{S'} = \bigcup_{n=1}^{\infty} \bigcup_{x \in S_n'} \left(x - \frac{1}{3}\, 2^{-n},\ x\right). \tag{7'}$$

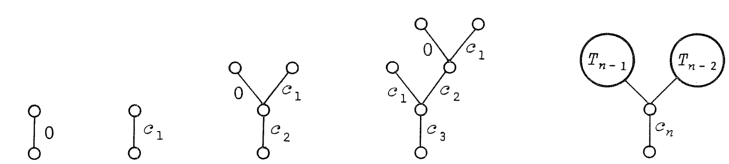
The intervals on the right of (7′) are disjoint because their endpoints belong to $\overline{S'}$, and their total length is

$$\sum_{n=1}^{\infty} \frac{1}{3}\, 2^{-n} u_{n-1} = \frac{2}{3},$$

which proves that $\overline{S'} = [0, 2/3] - C'$ has measure 0.

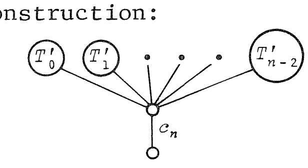
Equations (7) and (7′) emphasize the similarity between the constructions of $\overline{S}$ and $\overline{S'}$ and the construction of Cantor's ternary set. There are further similarities: $\overline{S}$ and $\overline{S'}$ are nowhere dense, $\overline{S'}$ is a perfect set, and the derived set $\overline{S}^*$ of $\overline{S}$ is also perfect.

The Fibonacci sequence of sets $(S_0, S_1, S_2, \ldots)$ may be represented graphically by a sequence of weighted, rooted trees $(T_0, T_1, T_2, \ldots)$ as follows:



T_0 $\quad T_1$ $\quad T_2$ $\quad T_3$ $\quad T_n\ (n \geqslant 2)$

For each of the u_n leaf-nodes of T_n, we may compute the total weight of the path to it from the root of T_n. The set of these u_n total weights is called "the shade of T_n" (cf. Turner [3]). The shade of T_n is obviously equal to the set S_n. A similar representation can be obtained for the dual of the Fibonacci sequence of sets by using the tree construction:

$T_0' = T_0$, $\quad T_1' = T_1$, $\quad T_n' =$  $\quad (n \geqslant 2)$

In particular, very pretty graphical illustrations of Theorems 1 and 1′ can be obtained (cf. Turner [3]).

REFERENCES

1. J. L. Brown, Jr. "A New Characterization of the Fibonacci Numbers." *The Fibonacci Quarterly* 3, no. 1 (1965):1-8.
2. C. G. Lekkerkerker. "Voorstelling van natuurlike getallen door een som van getallen van Fibonacci." *Simon Stevin* 29 (1952):190-195.
3. J. C. Turner. "Fibonacci Convolution Trees and Integer Representations." *New Zealand Mathematical Society Newsletter* 33 (1985):16-21.

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