LENGTH OF THE 7-NUMBER GAME

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1. INTRODUCTION

The n-number game is defined as follows. Let $S = (s_1, s_2, \ldots, s_n)$ be an n-tuple of nonnegative integers. Applying the difference operator D we obtain a new n-tuple $D(S) = (s_1', s_2', \ldots, s_n')$ by taking numerical differences; that is, $s_i' = |s_i - s_{i+1}|$. Subscripts are reduced modulo n so that $s_n' = |s_n - s_1|$. If this process is repeated, a sequence of tuples is generated; that is,

$$S, D^1(S), D^2(S), \ldots;$$

this sequence is referred to as *the n-number game generated by S*. The n-number game has been studied extensively, beginning with the 4-number game (see [5], [6], [8], [9], [13], and [19]).

As $k \Rightarrow \infty$, what happens to $D^k(S)$? Is it possible to generate an infinite, never repeating sequence? The answer is clearly no. For, let $|S| = \max(s_i)$. Then $|S| \geqslant |D(S)|$; since there are only a finite number of n-tuples with entries less than or equal to $|S|$, eventually the sequence $\{D^k(S)\}$ must repeat. When $n = 2^r$, it is well known that, for every S, the resulting sequence terminates with the zero-tuple $(0, 0, \ldots, 0)$. That this is not the case for other values of n is easily seen by considering the triple $S = (4, 5, 3)$. Applying the difference operator to this tuple gives the following:

$$\begin{aligned} S &= (4, 5, 3) \\ D^1(S) &= (1, 2, 1) \\ D^2(S) &= (1, 1, 0) \\ D^3(S) &= (0, 1, 1) \\ D^4(S) &= (1, 0, 1) \\ D^5(S) &= (1, 1, 0) = D^2(S) \end{aligned}$$

We call $\{D^2(S), D^3(S), D^4(S)\}$ a cycle. When n is not a power of 2, there are always tuples that appear in such cycles; indeed, for odd n, $(1, 1, 0, 0, \ldots, 0)$ is one such tuple. In an earlier paper this author characterized those tuples which can occur in a cycle. In particular, an n-tuple S is in a cycle only if all the entries in S are either 0 or $|S|$. Further, when n is odd, such tuples are in a cycle if and only if the number of nonzero entries is even [11].

For any n-tuple S, we say the game generated by S has *length* λ if $D^{\lambda}(S)$ is in a cycle but $D^{\lambda-1}(S)$ is not. We will denote the length of this game by $L(S)$; thus, in the example above, $L(S) = 2$. Note that we consider the zero-tuple to be in a cycle, namely the trivial one. There is no bound on the length of an n-number game. That is, for any λ there exists an n-tuple S such that $L(S) > \lambda$. We seek to characterize the upper bound of $L(S)$ for all tuples S for which $|S| \leqslant M$. Only when n equals 4 or $2^r + 1$ has this question been answered [18], [12]. In all other cases, only partial results are known; a complete solution seems very difficult. In this paper we will resolve the question for $n = 7$.

2. GENERAL RESULTS

There are $N = (M + 1)^n - M^n$ n-tuples with the property that $|S| = M$. However, we may consider several of these tuples related to each other and hence, in actuality, the number of tuples that we need consider is far less than N. For $S = (s_1, s_2, \ldots, s_n)$ with $|S| = M$, define $\mathfrak{M}(S)$ as the n-tuple given by

$$\mathfrak{M}(S) = (M - s_1, M - s_2, \ldots, M - s_n).$$

Further, let $\mathfrak{D}_n$ denote the dihedral group of a regular n-gon. Then we say two n-tuples *S and R are related* if either $R = \sigma(S)$ or $R = \sigma(\mathfrak{M}(S))$ for some $\sigma \in \mathfrak{D}_n$. If S and R are related, then we write $S \approx R$. It is easily seen that $\approx$ is an equivalence relation. We now show that related tuples have the same length. Thus, in determining those tuples which give games of maximum length, we need only consider the question up to equivalence classes.

Theorem 1: Suppose $S \approx R$, then $L(R) = L(S)$.

Proof: We may think of the entries of an n-tuple as the vertices of a regular n-gon. Thus, if $R = \sigma(S)$, R represents either a rotation, a flip, or a rotation followed by a flip of the n-gon whose vertices represent S. Clearly, the entries in $D(R)$ are the same as those in $D(S)$. It is only their order that is changed, and that change may be represented by a member of $\mathfrak{D}_n$. Since the entries are unchanged at each step, the length of R is the same as that of S.

Now, if $R = \mathfrak{M}(S)$, then it is easily seen that $D(S) = D(\mathfrak{M}(S))$ and hence the two tuples have the same length. ■

We will now turn our attention to those tuples which give games of maximum length. We first prove a lemma; the corollary that follows is then an immediate consequence.

Lemma 1: Let S be an n-tuple with $|S| = M$. Suppose that S has a predecessor and that S is not contained in a cycle. Then $|D^{n-2}(S)| \leqslant M - 1$.

Proof: If $S = (M, M, \ldots, M)$, then $|D(S)| = 0$ and thus the conclusion holds. Otherwise, since S has a predecessor and is not in a cycle, there exists some s_i that is different from 0 and M. Further, S has a predecessor if and only if there exist values $\delta_i \in \{-1, 1\}$ such that $\sum \delta_i s_i = 0$ [11]. Thus, S must have at least two entries that are different from 0 and M. This implies that $D(S)$ has at least three entries less than M; for, if $0 < s_i < M$, then $s'_{i-1} < M$ and $s'_i < M$, where $D(S) = (s'_1, s'_2, \ldots, s'_n)$. It follows that $D^{n-2}(S)$ has n entries less than M; i.e., $|D^{n-2}(S)| \leqslant M - 1$. ∎

Corollary 1: Let S be an n-tuple, $|S| = M$, then $L(S) \leqslant (n - 2)(M - 1) + 1$. ∎

This corollary gives an upper bound for $L(S)$ when $|S| \leqslant M$. For $n = 2^r + 1$, this upper bound is actually taken on by the n-tuple

$$R_M = (0, 0, \ldots, 0, M - 1, M) \text{ for } M \geqslant 1 \quad \text{(see [12])}.$$

For other values of n the actual bound is less than that found in Corollary 1. In general, for a given n, the least upper bound for $L(S)$ with $|S| = M$ is not known. The main result of this paper will be to characterize $L(S)$ when $n = 7$. In particular we will show that if S is a 7-tuple with $|S| \leqslant M$ and M is sufficiently large, then

$$L(S) \leqslant \begin{cases} 7(M - 1)/2 & \text{if } M \text{ is odd,} \\ 7(M - 2)/2 + 4 & \text{if } M \text{ is even.} \end{cases}$$

This can be fairly easily proved for tuples S for which $|D^7(S)| \leqslant M - 2$. Thus, we first determine the tuples for which $|D^7(S)| = M - 1$; note that by Lemma 1, $|D^7(S)| < M$.

3. TUPLES FOR WHICH $|D^7(S)| = M - 1$

In the following discussion we will only consider tuples up to the equivalence relation $\approx$. Thus, for example, when we state in Lemma 4 that

$$D^2(S) = (1, M, M, M, 1, 1, \cdot),$$

we really mean $D^2(S) \approx (1, M, M, M, 1, 1, \cdot)$.

By Lemma 1, if $|D^2(S)| = M - 1$, then $|D^7(S)| \leqslant M - 2$. Thus, in determining tuples for which $|D^7(S)| = M - 1$, we may restrict our attention to those with $|D^2(S)| = M$. First, consider an n-tuple $R = (r_1, r_2, \ldots, r_n)$ with $D(R) = (r'_1, r'_2, \ldots, r'_n)$. Suppose $D(R)$ has some entry $r'_i = M - 1$. Then either $r_i = M - 1$ and $r_{i+1} = 0$ or $r_i = M$ and $r_{i+1} = 1$. Thus, in the case under consideration, we must have, for some k, $2 \leqslant k \leqslant 6$,

$$|D^k(S)| = M, \quad D^k(S) \approx (M, 1, ., ., ., ., ., .),$$
$$D^{k+1}(S) \approx (M - 1, ., ., ., ., ., ., .), \text{ and } |D^7(S)| = M - 1.$$

For the tuple $D^{k+1}(S)$, we may have either $|D^{k+1}(S)| = M$ or $|D^{k+1}(S)| = M - 1$. However, what must happen is that $D^k(S)$ have enough consecutive M's and 1's to yield a sufficient number of $(M-1)$'s and 0's in $D^{k+1}(S)$ so that $D^7(S)$ contains at least one $M - 1$. Thus, it is the $M - 1$ and 0 terms in $D^{k+1}(S)$ that are the important ones. These possibilities are illustrated by the following two examples:

$$\begin{aligned}
S_1 &= (M, M, 0, M, 1, M, 2)\\
D^1(S_1) &= (0, M, M, M - 1, M - 1, M - 2, M - 2)\\
D^2(S_1) &= (M, 0, 1, 0, 1, 0, M - 2)\\
D^3(S_1) &= (M, 1, 1, 1, 1, M - 2, 2)\\
D^4(S_1) &= (M - 1, 0, 0, 0, M - 3, M - 4, M - 2)\\
D^5(S_1) &= (M - 1, 0, 0, M - 3, 1, 2, 1)\\
D^6(S_1) &= (M - 1, 0, M - 3, M - 4, 1, 1, M - 2)\\
D^7(S_1) &= (M - 1, M - 3, 1, M - 5, 0, M - 3, 1)
\end{aligned}$$

$$\begin{aligned}
S_2 &= (M, M, 0, M, 1, M - 2, 0)\\
D^1(S_2) &= (0, M, M, M - 1, M - 3, M - 2, M)\\
D^2(S_2) &= (M, 0, 1, 2, 1, 2, M)\\
D^3(S_2) &= (M, 1, 1, 1, 1, M - 2, 0)\\
D^4(S_2) &= (M - 1, 0, 0, 0, M - 3, M - 2, M)\\
D^5(S_2) &= (M - 1, 0, 0, M - 3, 1, 2, 1)\\
D^6(S_2) &= (M - 1, 0, M - 3, M - 4, 1, 1, M - 2)\\
D^7(S_2) &= (M - 1, M - 3, 1, M - 5, 0, M - 3, 1)
\end{aligned}$$

Note that in the examples above $|D^4(S_1)| = M - 1$, while $|D^4(S_2)| = M$. However, in both, it is the presence of five consecutive M's and 1's in $D^3(S_i)$ that gives rise to $|D^7(S_i)| = M - 1$. In general, then, if for a tuple S, we have $|S| = M$ and $|D^7(S)| = M - 1$, we will denote by $\kappa(S)$ that step where the presence of consecutive M's and 1's in $D^{\kappa(S)}(S)$ gives rise to $|D^7(S)| = M - 1$. Thus, in the examples above, $\kappa(S_1) = 3$ and $\kappa(S_2) = 3$. In general, we must have $2 \leqslant \kappa(S) \leqslant 6$. The following lemmas characterize $D^{\kappa(S)}(S)$ for various possibilities of $\kappa(S)$.

Lemma 2: Suppose S is a 7-tuple with the properties that $|S| = M$ and $|D^7(S)| = M - 1$. Let $k = \kappa(S)$ and let ℓ be the number of consecutive 1's and M's in $D^k(S)$, with ℓ as large as possible. Then $\ell \geqslant 8 - k$.

Proof: It is easily seen that the number of consecutive 0's and $(M - 1)$'s in $D^{k+1}(S)$ is $\ell - 1$ and, hence, $D^{k+1}(S)$ has at most $\ell - 1$ consecutive terms that equal $M - 1$. Similarly, the number of consecutive 0's and $(M - 1)$'s in $D^{k+t}(S)$ is $\ell - t$ and, hence, $D^{k+t}(S)$ has at most $\ell - t$ consecutive terms that equal

$M - 1$. Continuing, we find that $D^{k+\ell}(S)$ has no terms that equal $M - 1$. Since $|D^7(S)| = M - 1$, we must have $k + \ell \geqslant 8$ or $\ell \geqslant 8 - k$. ■

Lemma 3: Suppose S is a 7-tuple with the properties that $|S| = M$ and $|D^7(S)| = M - 1$. Let $k = \kappa(S)$. Then $D^k(S) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, a, \ldots, b)$, where a and b are neither 1 nor M and the number of consecutive 1's and M's is at least $8 - k$.

Proof: By Lemma 2, all that we need show is that $D^k(S)$ cannot have the form $(M, 1, \ldots, 1, M, c, \ldots, d)$. Suppose $D^k(S) = (M, 1, M, \ldots)$. Then $D^{k-1}(S)$ must equal $(0, M, M - 1, 2M - 1, \ldots)$ or $(M, 0, 1, M + 1, \ldots)$, both of which are impossible because $|S| = M$. Similarly, we find that $D^k(S)$ cannot equal $(M, 1, 1, 1, M, \ldots)$ or $(M, 1, 1, 1, 1, 1, M, \cdot)$.

Now, if $D^k(S) = (M, 1, 1, M, \ldots)$, then $D^{k-1}(S)$ must equal $(0, M, M - 1, M, 0, \ldots)$ or $(M, 0, 1, 0, M, \ldots)$, neither of which has a predecessor that contradicts the hypothesis. ■

The above lemmas mean that it is possible that there is a 7-tuple S for which $D^4(S) = (1, M, 1, 1, \ldots)$ and $|D^7(S)| = M - 1$. But there is no 7-tuple for which $D^3(S) = (1, M, 1, 1, a, \cdot, b)$, $a \neq 1$, $a \neq M$, $b \neq 1$, $b \neq M$, and $|D^7(S)| = M - 1$.

Lemma 4: Suppose S is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $\kappa(S) = 2$. Then one of the following must hold:

$$D^2(S) = (M, M, M, M, M, 1, \cdot) \qquad D^2(S) = (M, 1, 1, 1, 1, 1, \cdot)$$
$$D^2(S) = (1, M, 1, 1, 1, 1, \cdot) \qquad D^2(S) = (1, M, M, 1, 1, 1, \cdot)$$
$$D^2(S) = (1, M, M, M, 1, 1, \cdot)$$

Proof: It is easily verified that these five tuples give $|D^7(S)| = M - 1$. On the other hand, suppose R is a tuple such that $D^2(R) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, a)$. By direct computation, it can be shown that $|D^7(R)| < M - 1$; e.g., if $D^2(R) = (1, 1, M, 1, 1, 1, a)$, then $|D^7(R)| < M - 1$ unless $a = 1$. ■

Theorem 2: Suppose $M \geqslant 12$ and S is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $\kappa(S) = 2$. Then S is related to one of the following:

$$(1, 0, 0, M, M, 0, M - 1) \qquad (0, 13, 13, 12, 10, 7, 3)$$
$$(0, 15, 15, 14, 12, 9, 5)$$

Proof: We use Lemma 4, consider the various cases, and work our way backward to obtain S. We begin by assuming that $D^2(S) = (1, M, M, M, 1, 1, \cdot)$ and find possible tuples equal to $D(S)$ by first setting each of its elements equal, in turn, to M:

1.	M	$M - 1$	-1	$M - 1$	-1	0	1
2.	$M - 1$	M	0	M	0	1	0/2
3.	1	0	M	0	M	$M - 1$	$M - 2/M$
4.	same as Row 2						
5.	same as Row 3						
6.	0	-1	$M - 1$	-1	$M - 1$	M	$M - 1$
7.	same as Row 3						

When more than one number appears, such as "0/2" in Row 2, it means that either number is possible at that stage. Rows 1 and 6 are not possible because negative elements and present. We now treat Rows 2 and 3 in the above fashion. Starting with Row 3, we find possible tuples for S when $D(S) = (1, 0, M, 0, M, M - 1, M - 2)$, as follows:

1a.	M	$M + 1$	$M + 1$	1	1	$M + 1$	2
2a.	$M - 1$	M	M	0	0	M	1
3a.	same as Row 2a						
4a.	1	0	0	M	M	0	$M - 1$
5a.	same as Row 4a						
6a.	same as Row 2a						
7a.	2	1	1	$M + 1$	$M + 1$	1	M

Rows 1a and 7a are impossible because of the presence of elements greater than M. Rows 2a and 4a are possible, but they are related. Row 4a is the first tuple listed in the theorem. Continuing with Row 3, we find that there are no predecessors when $D(S) = (1, 0, M, 0, M, M - 1, M)$. Similarly, there are no predecessors for Row 2 if $M < 12$.

We repeat this process for each of the other four conditions in Lemma 4 and find that $D^2(S) = (M, 1, 1, 1, 1, 1, \cdot)$ gives rise to the other tuples for S listed in the theorem. ■

The other cases, with $2 < \kappa(S)$, proceed similarly.

Lemma 5: Suppose S is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $2 < \kappa(S) \leqslant 6$. Then one of the following must hold:

$$
\begin{array}{ll}
D^3(S) = (M, 1, 1, 1, 1, \cdot, \cdot) & D^3(S) = (M, M, 1, 1, 1, \cdot, \cdot) \\
D^3(S) = (M, M, M, 1, 1, \cdot, \cdot) & D^3(S) = (M, M, M, M, 1, \cdot, \cdot) \\
D^4(S) = (M, 1, 1, 1, \cdot, \cdot, \cdot) & D^4(S) = (M, M, M, 1, \cdot, \cdot, \cdot) \\
D^4(S) = (1, M, 1, 1, \cdot, \cdot, \cdot) & D^5(S) = (M, 1, 1, \cdot, \cdot, \cdot, \cdot) \\
D^5(S) = (M, M, 1, \cdot, \cdot, \cdot, \cdot) & D^6(S) = (M, 1, \cdot, \cdot, \cdot, \cdot, \cdot)
\end{array}
$$

Proof: It is easily verified that the above ten tuples give $|D^7(S)| = M - 1$. On the other hand, suppose R is a tuple not in the above list, such that $D^k(R) = (1, \ldots, 1, M, \ldots, M, 1, \ldots, 1, \ldots)$, where the number of consecutive 1's and M's is at least $8 - k$ and there is at least one 1. Then it must be the case that

$D^3(R) = (1, M, 1, 1, 1, \cdot, \cdot)$ $\quad$ $D^3(R) = (1, M, M, 1, 1, \cdot, \cdot)$

$D^3(R) = (1, M, M, M, 1, \cdot, \cdot)$ $\quad$ $D^3(R) = (1, 1, M, 1, 1, \cdot, \cdot)$

$D^4(R) = (M, M, 1, 1, \ , \cdot, \cdot)$ $\quad$ $D^4(R) = (1, M, M, 1, \cdot, \cdot, \cdot)$

$D^5(R) = (1, M, 1, \cdot, \cdot, \cdot, \cdot)$

or, more precisely, $D^k(S)$ must be related to one of these. By direct computation, it can be shown that $|D^7(R)| < M - 1$. ■

Theorem 3: Suppose $M \geqslant 12$ and S is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and $2 < \kappa(S) < 6$. Then S is related to one of the following tuples:

$D^3(S) = (M, 1, 1, 1, 1, \cdot, \cdot, \cdot)$:

$(0, M, 0, 0, M-1, M, 2)$ $\quad$ $(0, M, 0, 0, M-1, M-2, 0)$

$(0, 0, M, 0, M-1, 2, M-4)$ $\quad$ $(0, 0, M, 0, M-1, 2, M-2)$

$(0, 0, M, 0, M-1, 2, M)$ $\quad$ $(0, M, 0, 0, M-1, M, 0)$

$(0, 0, M, 0, M-1, 0, M-2)$ $\quad$ $(0, 0, M, 0, M-1, 0, M)$

$(0, 12, 12, 12, 11, 8, 2)$ $\quad$ $(0, 12, 12, 12, 11, 8, 4)$

$(0, 14, 14, 14, 13, 10, 4)$ $\quad$ $(0, 14, 14, 14, 13, 10, 6)$

$(0, 18, 18, 18, 17, 14, 8)$ $\quad$ $(0, 20, 20, 20, 19, 16, 10)$

$D^3(S) = (M, M, 1, 1, 1, \cdot, \cdot)$:

$(0, M, 0, 0, 0, 1, 2)$ $\quad$ $(0, M, 0, 0, 0, 1, 0)$

$(0, M, 0, 0, 1, 4)$

$D^3(S) = (M, M, M, 1, 1, \cdot, \cdot)$:

$(0, M, 0, 0, 0, M, M-1)$

$D^4(S) = (M, 1, 1, 1, \cdot, \cdot, \cdot)$:

$(0, 13, 13, 13, 13, 12, 8)$ $\quad$ $(0, 15, 15, 15, 15, 14, 10)$

$D^4(S) = (1, M, 1, 1, \cdot, \cdot, \cdot)$:

$(1, 0, 0, 0, 0, M, 1)$ $\quad$ $(M-1, 0, 0, M, M, M, 1)$

$(1, 0, M, 0, 0, M, M-1)$

$D^5(S) = (M, 1, 1, \cdot, \cdot, \cdot, \cdot)$:

$(0, M, 0, M, 0, 0, M-1)$ $\quad$ $(0, 0, 0, M, 0, M, M-1)$

$(0, M, M, M, 0, 0, 1)$ $\quad$ $(0, M, M, 0, M, M, M-1)$

$(0, 0, 0, 0, M, 0, 1)$ $\quad$ $(0, M, 0, 0, M, M, 1)$

Proof: The proof proceeds as in Theorem 2; due to the number of cases, the calculations are tedious. Although originally obtained by hand, these results were verified by computer. A copy of the program and/or output may be obtained from the author. ■

Theorem 4: Suppose S is related to $(1, 0, M, M, M, 0, 0)$. Then, for $M \geqslant 6$, $|D^{14}(S)| \leqslant M - 4$.

Proof: By direct calculation, we find $D^{12}(S) = (M-5, M-6, 1, 1, 2, 0, 1)$, so $|D^{12}(S)| \leqslant M - 4$ for $M \geqslant 6$ and thus $|D^{14}(S)| \leqslant M - 4$. ■

Theorem 5: Suppose S is a 7-tuple with the properties that $|S| = M$, $|D^7(S)| = M - 1$, and S is *not* related to $(1, 0, M, M, M, 0, 0)$. Then $|D^{10}(S)| \leqslant M - 3$ whenever $M \geqslant 12$.

Proof: Suppose that S is related to $(0, 1, M, M, 0, M, M)$. Computing $D^n(S)$ for $1 \leqslant n \leqslant 10$, we find $D^{10}(S) = (1, 1, M-6, M-4, 0, 1, 1)$ and thus the conclusion holds. Likewise, if S is related to $(0, 2, 8, 11, 12, 12, 12)$, then $D^{10}(S) = (5, 3, 3, 3, 2, 2, 2)$ and thus $|D^{10}(S)| \leqslant 9$. In a similar manner, the theorem can be verified by calculating $D^{10}(S)$ for each of the other twenty-nine tuples found in Theorems 2 and 3. ■

4. TUPLES WHICH GIVE LONGEST GAMES

Theorem 6: Let $T = (1, M, 1, M, M, 0, M)$ for $M \geqslant 1$. Then, for $M \geqslant 3$,

$$L(T_M) = \begin{cases} 7(M - 1)/2 & \text{if } M \text{ is odd,} \\ 7(M - 2)/2 + 4 & \text{if } M \text{ is even.} \end{cases}$$

Proof: For $M \geqslant 3$, it is easily seen by direct calculation that $D^7(T_M) = T_{M-2}$. Since $L(T_1) = 0$ and $L(T_2) = 4$, the result follows. ■

We will show that for $M \geqslant 8$ the tuples T_M, as defined above, give the games of maximum length. The following lemma is essentially a corollary of the previous theorem.

Lemma 6: For the tuples T_M defined as in Theorem 6, the following hold:

$$\begin{aligned} 7 + L(T_{M-2}) &= L(T_M) \\ 10 + L(T_{M-3}) &\leqslant L(T_M) \\ 14 + L(T_{M-4}) &= L(T_M) \end{aligned}$$

Proof: First suppose that M is even. Then we have:

$$\begin{aligned} 7 + L(T_{M-2}) &= 7 + 7(M - 4)/2 + 4 = 7(M - 2)/2 + 4 = L(T_M) \\ 10 + L(T_{M-3}) &= 10 + 7(M - 4)/2 \qquad = 7(M - 2)/2 + 3 < L(T_M) \\ 14 + L(T_{M-4}) &= 14 + 7(M - 6)/2 + 4 = 7(M - 2)/2 + 4 = L(T_M) \end{aligned}$$

When M is odd, the calculations are similar, except in that case

$$10 + L(T_{M-3}) = L(T_M). \ ■$$

Theorem 7: If $|S| = M$ and $M \geqslant 8$, then $L(S) \leqslant L(T_M)$.

Proof: It is easily verified by computer that the theorem holds for $M = 8, 9, 10$, and 11. This verification is not as lengthy as it might first appear. As noted above, we need only consider one member of each equivalence class. Further, note that if $S = (s_1, s_2, \ldots, s_n)$ with all $s_i \geqslant 1$, then $D(S) = D(T)$, where T is defined by $T = (s_1 - 1, s_2 - 1, \ldots, s_n - 1)$. Thus, we need only consider those tuples which have at least one zero entry. This significantly reduces the number of tuples that need to be checked.

We have shown that the theorem is true for all tuples S for which $|S| \leqslant m$ with $11 < m < M$. Consider a tuple S for which $|S| = M$ and $|D^7(S)| \leqslant M - 2$. Then we have, using Lemma 6,

$$L(S) \leqslant 7 + L(D^7(S)) \leqslant 7 + L(T_{M-2}) = L(T_M).$$

[Note that $L(S) = 7 + L(D^7(S))$ so long as $L(S) > 7$.] If $|D^7(S)| = M - 1$, then by Theorems 4 and 5, either $|D^{10}(S)| \leqslant M - 3$ or $|D^{14}(S)| \leqslant M - 4$. Thus, by induction and Lemma 6, either

$$L(S) \leqslant 10 + L(D^{10}(S)) \leqslant 10 + L(T_{M-3}) \leqslant L(T_M)$$

or

$$L(S) \leqslant 14 + L(D^{14}(S)) \leqslant 14 + L(T_{M-4}) = L(T_M)$$

whenever $M \geqslant 12$. ■

5. FURTHER QUESTIONS

Although showing that T_M gives a game of maximum length was not difficult, there were many details to consider. Additionally, there were many special cases for small values of M. This indicates that verifying an upper bound for the length of the general n-game is likely to be difficult. As stated above, for $n = 2^r + 1$, $r \geqslant 1$, games of maximum length are given by the tuples $(0, 0, \ldots, 0, M - 1, M)$ [12]. That is, games of maximum length arise from tuples with identical form. Whether this happens for other n is not known. For example, when $n = 2^r - 1$, $r \geqslant 4$, are games of maximum length given by tuples that are in some way similar in form to $(1, M, 1, M, M, 0, M)$?

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REFERENCES

1. E. R. Berlekamp. "The Design of Slowly Shrinking Labelled Squares." *Math. Comp.* *29* (1975):25-27.
2. K. D. Boklan. "The n-Number Game." *The Fibonacci Quarterly* *22* (1984):152-155.
3. M. Burmeister, R. Forcade, & E. Jacobs. "Circles of Numbers." *Glasgow Math. J.* *19* (1978):115-119.

4. C. Ciamberlini & A. Marengoni. "Su una Interessante Curiosita Numerica." *Period. Mat.* Ser. 4 **17** (1937):25-30.

5. M. Dumont & Jean Meeus. "The Four-Number Game." *J. Rec. Math.* **13** (1980-1981):89-96.

6. A. Ehrlich. "Columns of Differences." *Math Teaching* (1977):42-45.

7. B. Freedman. "The Four Number Game." *Scripta Math.* **14** (1948):35-47.

8. R. Honsberger. *Ingenuity in Mathematics*. New York: Random House, 1970.

9. H. H. Lammerich. "Quadrupelfolgen." *Praxis der Mathematik* **17** (1975):219-223.

10. M. Lotan. "A Problem in Difference Sets." *Amer. Math. Monthly* **56** (1949):219-223.

11. A. L. Ludington. "Cycles of Differences of Integers." *J. Number Theory* **13** (1981):155-261.

12. A. L. Ludington. "Length of the n-Number Game." (Submitted.)

13. L. Meyers. "Ducci's Four-Number Problem: A Short Bibliography." *Crux Mathematicorum* **8** (1982):262-266.

14. R. Miller. "A Game with n Numbers." *Amer. Math. Monthly* **85** (1978):183-185.

15. S. P. Mohanty. "On Cyclic Difference of Pairs of Integers." *Math. Stud.* **49** (1981):96-102.

16. D. Richman. "Iterated Absolute Differences." *Discrete Math.* (In press.)

17. W. Webb. "A Mathematical Curiosity." *Math. Notes from Wash. State Univ.* **20** (1980).

18. W. Webb. "The Length of the Four-Number Game." *The Fibonacci Quarterly* **20** (1982):33-35.

19. F. Wong. "Ducci Processes." *The Fibonacci Quarterly* **20** (1982):97-105.

20. P. Zvengrowski. "Iterated Absolute Differences." *Math. Magazine* **52** (1979):36-37.

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