ON SUMS OF THREE TRIANGULAR NUMBERS

JOHN A. EWELL
Northern Illinois University, DeKalb, IL 60115

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1. INTRODUCTION

According to Dickson [3, pp. 6 and 17], Fermat conjectured and Gauss proved the following theorem.

Theorem 1: Every nonnegative integer can be expressed as a sum of three triangular numbers [including $0 = 0(0 + 1)/2$].

Gauss also gave a method for counting the number of such representations of a given nonnegative integer. In this paper we propose to express the implicit counting function in terms of simple divisor functions. All of these functions are collected in the following definition.

Definition:

(i) For each nonnegative integer n, $t_3(n)$ denotes the cardinality of the set

$$\left\{(x_1, x_2, x_3) \in \mathbb{N}^3 \,\middle|\, n = \sum_{i=1}^{3} x_i(x_i + 1)/2\right\}.$$

(Here, $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

(ii) For each positive integer n and $i \in \{1, 5\}$,

$$d_i(n) := \sum_{\substack{\delta | n, \\ \delta \equiv i \ (\mathrm{mod}\ 6)}} 1;$$

and, $\varepsilon(n) := d_1(n) - d_5(n)$.

Theorem 2: Let n denote an arbitrary nonnegative integer.

(i) If $n = 3i(i + 1)/2$, for some $i \in \mathbb{N}$, then

$$t_3(n) = 1 + 3\sum_{i=0} \varepsilon(n - 3i(i + 1)/2).$$

(ii) If n is not of the form $3i(i + 1)/2$, then

$$t_3(n) = 3\sum_{i=0} \varepsilon(n - 3i(i + 1)/2).$$

In both cases, summation extends over all $i \in \mathbb{N}$ for which $n - 3i(i + 1)/2 > 0$.

Section 2 is dedicated to proof of Theorem 2. In view of the two theorems we then deduce a corollary concerning the behavior of the function ε.

2. PROOF OF THEOREM 2

The leading role in our argument is played by the following variant of the quintuple-product identity.

$$\prod_1^\infty \frac{(1 - x^{2n})(1 - a^2x^{2n-2})(1 - a^{-2}x^{2n})}{(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1})} = \sum_{-\infty}^{\infty} x^{n(3n+2)}(a^{-3n} - a^{3n+2}). \tag{1}$$

(Here and throughout our discussion we assume that a and x denote complex numbers with $a \neq 0$ and $|x| < 1$.) For a discussion of (1) and other forms of the quintuple-product identity see [5]. We shall also require the classical triple-product identity:

$$\prod_1^\infty (1 - x^{2n})(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}a^n. \tag{2}$$

In [2] Carlitz and Subbarao show how to deduce one form of the quintuple-product identity from (2).

Multiplying (1) by a^{-1}, we have

$$(a - a^{-1}) \prod_1^\infty \frac{(1 - x^{2n})(1 - a^2x^{2n})(1 - a^{-2}x^{2n})}{(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1})} \tag{3}$$

$$= a \sum_{-\infty}^{\infty} x^{3n^2+2n}a^{3n} - a^{-1} \sum_{-\infty}^{\infty} x^{3n^2+2n}a^{-3n}$$

$$= a \prod_1^\infty (1 - x^{6n})(1 + a^3x^{6n-1})(1 + a^{-3}x^{6n-5})$$

$$- a^{-1} \prod_1^\infty (1 - x^{6n})(1 + a^{-3}x^{6n-1})(1 + a^3x^{6n-5}).$$

In the last step we have used (2) to transform the infinite series into infinite products. For the sake of brevity, put

$$F(a) = F(a, x) := \prod_1^\infty \frac{(1 - a^2x^{2n})(1 - a^{-2}x^{2n})}{(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1})}$$

$$G(a) = G(a, x) := \prod_1^\infty (1 + a^3x^{6n-1})(1 + a^{-3}x^{6n-5}),$$

$$H(a) := G(a^{-1}).$$

Hence, (3) becomes

$$\prod_1^\infty (1 - x^{2n})(a - a^{-1})F(a) = \prod_1^\infty (1 - x^{6n})\{aG(a) - a^{-1}H(a)\}.$$

Differentiating the foregoing identity with respect to a, we get

$$\prod_1^\infty (1 - x^{2n})\{(1 + a^{-2})F(a) + (a - a^{-1})F'(a)\} \tag{4}$$

$$= \prod_1^\infty (1 - x^{6n})\{G(a) + a^{-2}H(a) + aG'(a) - a^{-1}H'(a)\}.$$

Now, using the technique of logarithmic differentiation, we evaluate $G'(a)$ and $H'(a)$, then substitute these evaluations into (4), let $a \to 1$, and cancel a factor of 2 in the resulting identity to get

$$\prod_1^\infty \frac{(1 - x^{2n})^3}{(1 + x^{2n-1})^2}$$

$$= \prod_1^\infty (1 - x^{6n})(1 + x^{6n-1})(1 + x^{6n-5})\left\{1 + 3\sum_1^\infty\left(\frac{x^{6n-1}}{1 + x^{6n-1}} - \frac{x^{6n-5}}{1 + x^{6n-5}}\right)\right\}.$$

In the foregoing identity we then let $x \to -x$, utilize the definition of ε, and simplify to get

$$\prod_1^\infty \frac{(1 - x^{2n})^3 \cdot (1 - x^{6n-3})}{(1 - x^{2n-1})^3 \cdot (1 - x^{6n})} = 1 + 3\sum_1^\infty \varepsilon(n)x^n. \tag{5}$$

At this juncture, we appeal to the following well-known identity of Gauss [4, p. 284].

$$\prod_1^\infty \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_0^\infty x^{n(n+1)/2}.$$

Hence, (5) becomes

$$\left\{\sum_0^\infty x^{n(n+1)/2}\right\}^3 = \sum_0^\infty x^{3n(n+1)/2}\left\{1 + 3\sum_1^\infty \varepsilon(n)x^n\right\},$$

or, equivalently (owing to the fact that the left side of this identity generates t_3),

$$\sum_0^\infty t_3(n)x^n = \sum_{i=0}^\infty x^{3i(i+1)/2} + 3\sum_{n=1}^\infty x^n \sum_{i=0} \varepsilon(n - 3i(i + 1)/2).$$

Equating coefficients of like powers of x, we thus prove our theorem.

Corollary: If n is any positive integer which is not of the form $3i(i + 1)/2$, then there exists $j \in \{0, 1, \ldots, [(-1 + \sqrt{(8/3)n + 1})/2]\}$ such that

$$\varepsilon(n - 3j(j + 1)/2) > 0.$$

Proof: Let such an n be given. By multiplicative induction it follows easily that $\varepsilon(m) \geqslant 0$ for each positive integer m. Hence, the sum on the right side of the equation of Theorem 2(ii) is nonnegative. Now, by Theorem 1, $t_3(n) > 0$. Hence, the aforementioned sum is positive, whence there exists

$$j \in \{0, 1, \ldots, [(-1 + \sqrt{(8/3)n + 1})/2]\}$$

such that

$$\varepsilon(n - 3j(j + 1)/2) > 0.$$

CONCLUDING REMARKS

In a recent paper, Andrews [1] has presented a proof of Theorem 1 which (unlike Gauss's proof) is independent of the theory of ternary quadratic forms. Of course, such proofs of Theorem 1 and Theorem 2 then combine to yield a proof of the Corollary that is independent of the theory of ternary quadratic forms. However, if one could find another such *direct* proof of the Corollary, then one could use the statement of the Corollary (then independent of Theorems 1 and 2) and Theorem 2 to produce yet another proof of Theorem 1.

REFERENCES

1. G. E. Andrews. "EYPHKA! Num = Δ + Δ + Δ." *J. of Number Theory* **23**, no. 3 (2986):285-293.
2. L. Carlitz & M. V. Subbarao. "A Simple Proof of the Quintuple-Product Identity." *Proc. Amer. Math. Soc.* **32** (1972):42-44.
3. L. E. Dickson. *History of the Theory of Numbers*, II. New York: Chelsea, 1952.
4. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*, 4th ed. Oxford: Oxford University Press, 1960.
5. M. V. Subbarao & M. Vidyasagar. "On Watson's Quintuple-Product Identity." *Proc. Amer. Math. Soc.* **26** (1970):23-27.

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