ON A GENERALIZATION OF THE FIBONACCI SEQUENCE IN THE CASE OF THREE SEQUENCES

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A new direction for generalizing the Fibonacci sequence was introduced in [1], and [2]. In this paper, we shall continue that direction of research.

Let $C_1, C_2, \ldots, C_6$ be fixed real numbers. Using C_1 to C_6, we shall construct new schemes which are of the Fibonacci type and are called 3-F-sequences. Our analogy is of [1] and [2]; the form is

$$\begin{cases} a_0 = C_1,\ b_0 = C_2,\ c_0 = C_3,\ a_1 = C_4,\ b_1 = C_5,\ c_1 = C_6 \\ a_{n+2} = x^1_{n+1} + y^1_n \\ b_{n+2} = x^2_{n+1} + y^2_n \\ c_{n+2} = x^3_{n+1} + y^3_n \end{cases} \quad (n \ge 0),$$

where $\langle x^1_{n+1}, x^2_{n+1}, x^3_{n+1}\rangle$ is any permutation of $\langle a_{n+1}, b_{n+1}, c_{n+1}\rangle$ and $\langle y^1_n, y^2_n, y^3_n\rangle$ is any permutation of $\langle a_n, b_n, c_n\rangle$.

The number of different schemes is obviously 36.

In [3], the specific scheme

$$\begin{cases} a_0 = C_1,\ b_0 = C_2,\ c_0 = C_3,\ a_1 = C_4,\ b_1 = C_5,\ c_1 = C_6 \\ a_{n+2} = b_{n+1} + c_n \\ b_{n+2} = c_{n+1} + a_n \\ c_{n+2} = a_{n+1} + b_n \end{cases} \quad (n \ge 0),$$

is discussed in detail. For the sake of brevity, we devise the following representation for this scheme:

$$S = \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix}. \tag{1}$$

Note that we have merely eliminated the subscripts and the equal and plus symbols so that our notation is similar to that used in representing a system of linear equations in matrix form. Using this notation, it is important to remember that the elements in their first column are always in the same order while the elements in the other column can be permuted within that column. Every element a, b, and c must be used in each column.

We now define an operation called substitution over these 3-F-sequences and adopt the notation $[p, q]S$, where $p, q \in \{a, b, c\}$, $p \ne q$. Applying the operation to S merely interchanges all occurrences of p and q in each column. For example, using (1), we have

$$[a, c]S = \begin{Bmatrix} c & b & a \\ b & a & c \\ a & c & b \end{Bmatrix}. \tag{2}$$

Note that in the result we do not maintain the order of the elements in the first column. To maintain this order we interchange the first and last rows of [2] to obtain

$$S' = \begin{Bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{Bmatrix},$$

which corresponds to the scheme

$$\begin{cases} a_0 = C_1',\ b_0 = C_2',\ c_0 = C_3',\ a_1 = C_4',\ b_1 = C_5',\ c_1 = C_6' \\ a_{n+2} = c_{n+1} + b_n \\ b_{n+2} = a_{n+1} + c_n \qquad (n \geq 0), \\ c_{n+2} = b_{n+1} + a_n \end{cases}$$

where C_1', C_2', ..., C_6' are real numbers.

We shall say that the two schemes S and S' are equivalent under the operation of substitution and denote this by

$$S \leftrightarrow S'.$$

It is now obvious that for any two 3-F-sequences S and S', if $[p, q]S \leftrightarrow S'$, then $[p, q]S' \leftrightarrow S$. To investigate the concept of equivalence to a deeper extent, it is necessary to list all 36 schemes:

$$S_1 = \begin{Bmatrix} a & a & a \\ b & b & b \\ c & c & c \end{Bmatrix} \quad S_2 = \begin{Bmatrix} a & a & a \\ b & b & c \\ c & c & b \end{Bmatrix} \quad S_3 = \begin{Bmatrix} a & a & a \\ b & c & b \\ c & b & c \end{Bmatrix} \quad S_4 = \begin{Bmatrix} a & a & a \\ b & c & c \\ c & b & b \end{Bmatrix}$$

$$S_5 = \begin{Bmatrix} a & a & b \\ b & b & a \\ c & c & c \end{Bmatrix} \quad S_6 = \begin{Bmatrix} a & a & b \\ b & b & c \\ c & c & a \end{Bmatrix} \quad S_7 = \begin{Bmatrix} a & a & b \\ b & c & a \\ c & b & c \end{Bmatrix} \quad S_8 = \begin{Bmatrix} a & a & b \\ b & c & c \\ c & b & a \end{Bmatrix}$$

$$S_9 = \begin{Bmatrix} a & a & c \\ b & b & a \\ c & c & b \end{Bmatrix} \quad S_{10} = \begin{Bmatrix} a & a & c \\ b & b & b \\ c & c & a \end{Bmatrix} \quad S_{11} = \begin{Bmatrix} a & a & c \\ b & c & a \\ c & b & b \end{Bmatrix} \quad S_{12} = \begin{Bmatrix} a & a & c \\ b & c & b \\ c & b & a \end{Bmatrix}$$

$$S_{13} = \begin{Bmatrix} a & b & a \\ b & a & b \\ c & c & c \end{Bmatrix} \quad S_{14} = \begin{Bmatrix} a & b & a \\ b & a & c \\ c & c & b \end{Bmatrix} \quad S_{15} = \begin{Bmatrix} a & b & a \\ b & c & b \\ c & a & c \end{Bmatrix} \quad S_{16} = \begin{Bmatrix} a & b & a \\ b & c & c \\ c & a & b \end{Bmatrix}$$

$$S_{17} = \begin{Bmatrix} a & b & b \\ b & a & a \\ c & c & c \end{Bmatrix} \quad S_{18} = \begin{Bmatrix} a & b & b \\ b & a & c \\ c & c & a \end{Bmatrix} \quad S_{19} = \begin{Bmatrix} a & b & b \\ b & c & a \\ c & a & c \end{Bmatrix} \quad S_{20} = \begin{Bmatrix} a & b & b \\ b & c & c \\ c & a & a \end{Bmatrix}$$

$$S_{21} = \begin{Bmatrix} a & b & c \\ b & a & a \\ c & c & b \end{Bmatrix} \quad S_{22} = \begin{Bmatrix} a & b & c \\ b & a & b \\ c & c & a \end{Bmatrix} \quad S_{23} = \begin{Bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{Bmatrix} \quad S_{24} = \begin{Bmatrix} a & b & c \\ b & c & b \\ c & a & a \end{Bmatrix}$$

$$S_{25} = \begin{Bmatrix} a & c & a \\ b & a & b \\ c & b & c \end{Bmatrix} \quad S_{26} = \begin{Bmatrix} a & c & a \\ b & a & c \\ c & b & b \end{Bmatrix} \quad S_{27} = \begin{Bmatrix} a & c & a \\ b & b & b \\ c & a & c \end{Bmatrix} \quad S_{28} = \begin{Bmatrix} a & c & a \\ b & b & c \\ c & a & b \end{Bmatrix}$$

$$S_{29} = \begin{Bmatrix} a & c & b \\ b & a & a \\ c & b & c \end{Bmatrix} \quad S_{30} = \begin{Bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{Bmatrix} \quad S_{31} = \begin{Bmatrix} a & c & b \\ b & b & a \\ c & a & c \end{Bmatrix} \quad S_{32} = \begin{Bmatrix} a & c & b \\ b & b & c \\ c & a & a \end{Bmatrix}$$

$$S_{33} = \begin{Bmatrix} a & c & c \\ b & a & a \\ c & b & b \end{Bmatrix} \quad S_{34} = \begin{Bmatrix} a & c & c \\ b & a & b \\ c & b & a \end{Bmatrix} \quad S_{35} = \begin{Bmatrix} a & c & c \\ b & b & a \\ c & a & b \end{Bmatrix} \quad S_{36} = \begin{Bmatrix} a & c & c \\ b & b & b \\ c & a & a \end{Bmatrix}$$

Note that $S = S_{23}$ and $S' = S_{30}$, so that $S_{23} \leftrightarrow S_{30}$.

We say that a 3-F-sequence S is trivial if at least one of the resulting sequences is a Fibonacci sequence. Otherwise, S is said to be an Essential Generalization of the Fibonacci sequence.

Observe that there are ten trivial 3-F-sequences. They are

$$S_1,\ S_2,\ S_3,\ S_4,\ S_5,\ S_{10},\ S_{13},\ S_{17},\ S_{27},\ S_{36}.$$

These 10 schemes are easy to detect since they have at least one row all with the same letter. Furthermore, for these schemes one of the three possible substitutions returns the scheme itself. For example,

$$[b,\ c]S_i \leftrightarrow S_i,\ i = 1,\ 2,\ 3,\ 4$$
$$[a,\ b]S_i \leftrightarrow S_i,\ i = 1,\ 5,\ 13,\ 17$$
$$[a,\ c]S_i \leftrightarrow S_i,\ i = 1,\ 10,\ 27,\ 36.$$

The twenty-six remaining schemes are Essential Generalizations of the Fibonacci sequence. For eight of these schemes, the result is independent of the substitution made. That is,

$$[p,\ q]S_6 \leftrightarrow S_9 \qquad [p,\ q]S_{15} \leftrightarrow S_{25}$$
$$[p,\ q]S_{20} \leftrightarrow S_{33} \qquad [p,\ q]S_{23} \leftrightarrow S_{30}$$

for all p and q. This means the substitution operation for these schemes is cyclic of length 2.

For the other eighteen Essential Generalizations of the Fibonacci sequence schemes, all three possible substitutions generate three different schemes. For example,

$$[a,\ b]S_7 \leftrightarrow S_{31} \qquad [a,\ b]S_8 \leftrightarrow S_{35} \qquad [a,\ b]S_{16} \leftrightarrow S_{34}$$
$$[a,\ c]S_7 \leftrightarrow S_{14} \qquad [a,\ c]S_8 \leftrightarrow S_{21} \qquad [a,\ c]S_{16} \leftrightarrow S_{29}$$
$$[b,\ c]S_7 \leftrightarrow S_{12} \qquad [b,\ c]S_8 \leftrightarrow S_{11} \qquad [b,\ c]S_{16} \leftrightarrow S_{26}$$

All of the substitutions associated with the remaining eighteen schemes and their results are conveniently illustrated by the following three figures. That is, these pictures determine all possible cycles.

For example,

$$[a,\ b]S_{29} \leftrightarrow S_{19}$$

and

$$[a,\ b]([b,\ c]([a,\ c]S_{24})) \leftrightarrow S_{29}.$$

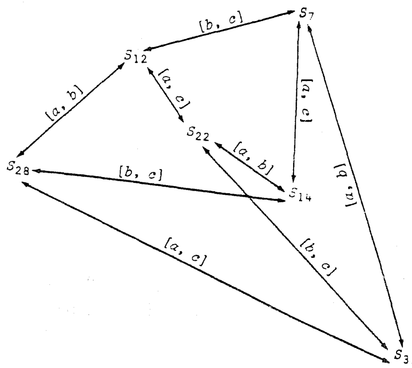

FIGURE 1

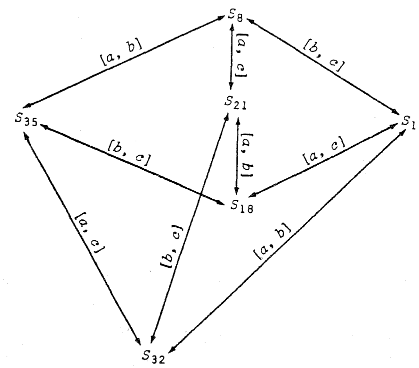

FIGURE 2

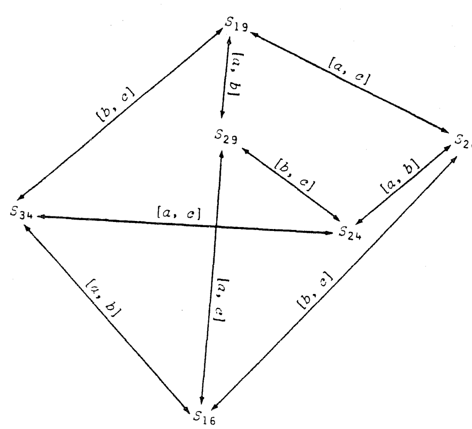

FIGURE 3

Note that the figures tell us that many of the schemes are independent. That is, S_{24} and S_{18} are independent. In fact, S_{18} is related only to the six schemes listed in Figure 3. Similar results can be found for the other schemes.

The closed form equation of the members for all three sequences of scheme S_{23} is given in [3]. By a method similar to that given in [3] or [1], the closed form equation of the members for all three sequences of the other schemes can be determined. We leave this task to the reader. Obviously, these results could be generalized to the case of four or more sequences with very little difficulty.

References

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