ON A CERTAIN SEQUENCE OF QUOTIENTS OF A SEQUENCE

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Let $\{x_n\}_{n \geq 1}$ be a sequence of positive real numbers. The q-sequence corresponding to $\{x_n\}$ is defined as having as its n^{th} term

$$q_{n+1} = \frac{x_{n+1}}{x_n}$$

for all integers $n \geq 1$. One of the purposes of this note is to compare the sequence $\{x_n\}$ with its corresponding q-sequence $\{q_n\}$ so that conditions imposed on one of them will yield results concerning the other.

1. *Example:* Consider the Fibonacci sequence $\{x_n\}$ defined recursively by

$$x_1 = 1,\ x_2 = 1,\ x_{n+1} = x_n + x_{n-1} \text{ if } n \geq 2.$$

It is well known that the q-sequence corresponding to $\{x_n\}$ converges to the real number $(1 + \sqrt{5})/2$. This example shows that divergent sequences $\{x_n\}$ can have corresponding q-sequences that converge. On the other hand, examples can be found of convergent sequences of quotients of convergent sequences. See Theorem 5 below.

Whenever a sequence $\{x_n\}$ is defined recursively, say

$$x_1 = a,\ x_2 = b,\ x_{n+1} = f(x_n,\ x_{n-1}) \text{ for } n \geq 2$$

and positive numbers a and b, let $S(a, b, f)$ denote the corresponding q-sequence, where f is a nonnegative function of two real variables which is defined and positive in the first quadrant and defined on the positive y-axis. If $\{q_n\}$ converges, let z be its limit

$$z = \lim_{n \to \infty} q_n.$$

2. *Theorem:* Let $\{q_n\} = S(a, b, f)$ be the q-sequence corresponding to a sequence $\{x_n\}$. If $\{q_n\}$ converges and f is continuous and positively homogeneous of degree 1 $[f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda > 0]$, then z satisfies the equation

$$w^2 = f(w, 1).$$

Proof: Since f is positively homogeneous of degree 1, it follows that

$$q_{n+1}q_n = \frac{x_{n+1}}{x_{n-1}} = \frac{f(x_n,\ x_{n-1})}{x_{n-1}} = f\left(\frac{x_n}{x_{n-1}},\ 1\right) = f(q_n,\ 1).$$

Consequently, $z^2 = f(z, 1)$ must hold because of the continuity of f.

3. *Examples:* (1) For the Fibonacci sequence, one has

$$f(x_n,\ x_{n-1}) = x_n + x_{n-1}$$

and the limit $z = (1 + \sqrt{5})/2$ satisfies $f(z, 1) = ((1 + \sqrt{5})/2)^2$, in agreement with the theorem.

(2) Consider $\{q_n\} = S(1, 2, x_n + 2x_{n-1})$. To find z, one might want first to solve the quadratic equation $z^2 = z + 2$, whose positive solution is $z = 2$. Unfortunately, Theorem 2 as stated does not guarantee that $\lim_{n\to\infty} q_n = 2$. If the limit exists, then

$$q_{n+1} = \frac{x_{n+1}}{x_n} = \frac{x_n + 2x_{n-1}}{x_n} = 1 + 2\Big/\left(\frac{x_n}{x_{n-1}}\right) = 1 + 2/q_n$$

implies $z = 2$. A procedure for finding the limit is presented in the next result.

4. *Theorem:* Let $b > 0$ and $c \geq 0$ be real numbers. If $f(x, y) = bx + cy$, let $\{x_n\}$ be the sequence defined recursively by

$$x_1 = p > 0,\ x_2 = q > 0, \text{ and } x_{n+1} = f(x_n, x_{n-1}) \text{ for } n \geq 2.$$

Then the q-sequence $S(p, q, f)$ converges to

$$z = (b + \sqrt{b^2 + 4c})/2$$

independent of the initial values p and q. Moreover, the sequence $\{q_{n+1} - q_n\}$ is either the constant sequence $\{0\}$ or oscillates between positive and negative values.

Proof: Note that for $n \geq 2$,

$$q_{n+1} = \frac{x_{n+1}}{x_n} = \frac{bx_n + cx_{n-1}}{x_n} = b + \frac{c}{q_n},$$

and hence, for $n \geq 3$,

$$q_{n+1} - q_n = c\,\frac{q_{n-1} - q_n}{q_n q_{n-1}}.$$

Consequently, $\{q_{n+1} - q_n\}$ is either the sequence $\{0\}$ or oscillates between positive and negative values. Also,

$$q_n = b + \frac{c}{q_{n-1}} \text{ for } n \geq 3$$

implies that $q_n q_{n-1} = bq_{n-1} + c > b^2 + c$ for $n \geq 4$, and hence

$$|q_{n+1} - q_n| = \left|c\,\frac{q_{n-1} - q_n}{q_n q_{n-1}}\right| \leq \frac{c}{b^2 + c}|q_{n-1} - q_n|.$$

If $d = c/(b^2 + c)$, then $0 \leq d < 1$ and

$$|q_{n+1} - q_n| \leq d^{n-3}|q_4 - q_3| \text{ for } n \geq 3.$$

Since

$$\sum_{n=3}^{\infty} d^{n-3}|q_4 - q_3|$$

converges, it follows that $\{q_n\}$ is a Cauchy sequence, thus it converges to some number $z \geq 0$. Theorem 2 shows that $z^2 = bz + c$ must hold; therefore,

$$z = \frac{b + \sqrt{b^2 + 4c}}{2}. \square$$

One can, in some cases, compare the behavior of a given sequence $\{x_n\}$ with that of its corresponding q-sequence. It is to be pointed out here that the sequences referred to in the following result are not necessarily generated by recursion

5. *Theorem:* Let $\{x_n\}$ be a sequence of positive numbers and let $\{q_n\}$ denote its corresponding q-sequence. Then

(1) If $\{q_n\} \in c_0$, then $\{x_n\} \in \ell^{(1)}$; hence $\{x_n\} \in c_0$.

(2) If $\{x_n\} \in c - c_0$, then $\lim\limits_{n\to\infty} q_n = 1$.

($\ell^{(1)}$, c_0, and c denote the spaces of summable, convergent to zero, and convergent sequences, respectively.)

6. *Example:* Consider the sequence defined by

$$y_n = \frac{1}{4n^2 - 1}.$$

Since $4n^2 - 1 > n^2$, then $\{y_n\} \in \ell^{(1)}$. Define a sequence $\{x_n\}$ by

$$x_n = \frac{t}{\prod\limits_{k \ge n} (1 + y_k)},$$

where $t > 0$ is a real parameter. This sequence is well defined, because the infinite product

$$\prod_{k \ge n} (1 + y_k)$$

converges. It follows that $\{1 + y_n\}$ is a q-sequence. Indeed,

$$\frac{x_{n+1}}{x_n} = \frac{\prod\limits_{k \ge n} (1 + y_k)}{\prod\limits_{k \ge n+1} (1 + y_k)}$$

for $n \ge 1$. A simple computation shows that

$$\prod_{k \ge 1} (1 + y_k) = \frac{2}{1}\,\frac{2}{3}\,\frac{4}{3}\,\frac{4}{5}\,\frac{6}{5}\,\frac{6}{7}\,\cdots,$$

where the product on the right was shown by John Wallis (1616-1703) to have the value $\pi/2$.

The next result is an attempt to answer a question suggested by the previous example: What sequences are q-sequences?

7. *Theorem:* Let $\{y_n\}$ be a sequence of positive terms in $\ell^{(1)}$. Then, there exists a 1-parameter family of sequences $\{x_n(t)\}$ for $t > 0$ such that

$$\frac{x_{n+1}(t)}{x_n(t)} = 1 + y_n$$

for all $n \ge 1$.

Proof: In order to have that $x_{n+1}/x_n = 1 + y_n$, one must solve the infinite linear system

$$\begin{cases} x_1(1 + y_1) - x_2 = 0 \\ x_2(1 + y_2) - x_3 = 0 \\ \quad\vdots \end{cases}$$

Set $x_1 = t$, an arbitrary positive real number. Then $x_2 = t(1 + y_1)$, $x_3 = t(1 + y_1)(1 + y_2)$, and by induction,

$$x_{n+1} = t \prod_{k \le n} (1 + y_k).$$

Therefore, the q-sequence corresponding to $\{x_n(t)\}$ is given by

$$\frac{x_{n+1}(t)}{x_n(t)} = \frac{t \prod_{k \le n} (1 + y_k)}{t \prod_{k \le n-1} (1 + y_k)} = 1 + y_n.$$

This establishes the result.

In Theorem 2 above, the limit z of a convergent q-sequence corresponding to a recursively generated sequence was shown to satisfy a functional equation involving the generating function for the original sequence. This generating function was required to be positively homogeneous of degree 1, continuous and nonnegative in the first quadrant. According to Theorem 4, if the generating function is the restriction of a linear form, then the limit of the q-sequence can be explicitly calculated and does not depend on the initial two terms of the original sequence.

The result that follows explores the nature of the functional equation by characterizing the class of functions to which Theorem 2 applies and provides examples to show that the independence of the limit z of a q-sequence with respect to the initial terms p and q of the original sequence, which was one of the conclusions obtained in Theorem 4, no longer holds in the general case.

8. *Theorem:* A function $f: [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous, positive on $(0, \infty)^2$, and positively homogeneous of degree 1 if and only if there is a continuous function $\gamma: (0, \infty) \to (0, \infty)$ which is such that

(i) $\gamma(t) = f(1, t)$ for all $t \in (0, \infty)$

and

(ii) $\lim_{t \to \infty} \frac{\gamma(t)}{t}$ exists and is finite.

Proof: Suppose $f: [0, \infty) \times (0, \infty) \to [0, \infty)$ is continuous, positive on $(0, \infty)^2$ and positively homogeneous of degree 1. Set

$$\gamma(t) = f(1, t) \text{ for } t \in (0, \infty).$$

Then γ is continuous and positive, and

$$\lim_{t \to \infty} \frac{\gamma(t)}{t} = \lim_{t \to \infty} f\left(\frac{1}{t}, 1\right) = f(0, 1)$$

exists and is finite, due to the continuity of f and the fact that f is positively homogeneous of degree 1.

Conversely, suppose that $\gamma: (0, \infty) \to (0, \infty)$ is continuous and that

$$\lim_{t \to \infty} \frac{\gamma(t)}{t}$$

exists and is finite. Set

$$\alpha = \lim_{t \to \infty} \frac{\gamma(t)}{t}$$

and define f on $[0, \infty) \times (0, \infty)$ by setting

$$f(x, y) = \begin{cases} x\gamma(y/x) & \text{if } x \neq 0 \\ y\alpha & \text{if } x = 0. \end{cases}$$

f is continuous for $x > 0$ and $y > 0$ since is continuous on $(0, \infty)$, $(x, y) \to y/x$ is continuous for $x > 0$ and $y > 0$ and the projection $(x, y) \to x$ is continuous everywhere.

If $S_1 = \{(x, y) : x > 0,\ y > 0\}$, $S_2 = \{(0, y) : y > 0\}$ and if y_0 is a fixed positive number, then

$$\lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} f(x, y) = \lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} x\gamma\left(\frac{y}{x}\right) = \lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_1}} y\left[\frac{\gamma\left(\frac{y}{x}\right)}{\frac{y}{x}}\right] = y_0\alpha = f(0, y_0),$$

and $$\lim_{\substack{(x, y) \to (0, y_0) \\ (x, y) \in S_2}} f(x, y) = \lim_{y \to y_0} \alpha y = y_0\alpha = f(0, y_0).$$

Thus,

$$\lim_{(x, y) \to (0, y_0)} f(x, y) = f(0, y_0),$$

and f is continuous at $(0, y_0)$. It has now been demonstrated that f is continuous on $[0, \infty) \times (0, \infty)$.

The function f is also positively homogeneous of degree 1. For, if $\lambda > 0$,

$$f(\lambda x, \lambda y) = \begin{cases} (\lambda x)\gamma\left(\frac{\lambda y}{\lambda x}\right) & \text{if } x \neq 0 \\ (\lambda y)\alpha & \text{if } x = 0 \end{cases} = \begin{cases} \lambda\ x\gamma\left(\frac{y}{x}\right) & \text{if } x \neq 0 \\ \lambda(y\alpha) & \text{if } x = 0. \end{cases}$$

Therefore, $f(\lambda x, \lambda y) = \lambda f(x, y)$. □

Since $x_{n+1} = f(x_n, x_{n-1})$ implies

$$q_{n+1} = f\left(1, \frac{1}{q_n}\right) = \gamma\left(\frac{1}{q_n}\right),$$

the question of the convergence of q-sequences is equivalent to examining the convergence of sequences $\{q_n\}$ generated by choosing $q_2 > 0$ and defining q_n for $n \geq 3$ by

$$q_n = \gamma\left(\frac{1}{q_{n-1}}\right)$$

for some positive continuous function γ on $(0, \infty)$ having the property that

$$\lim_{t \to \infty} \frac{\gamma(t)}{t}$$

exists and is finite. Using this fact, examples can be constructed quite easily. The following examples, which were constructed in this way, show that limits of q-sequences depend in general on the starting values x_1 and x_2.

9. *Example:* Let $\gamma(t) = 1/t^2$. Starting with $q_2 = 1$, it follows that $q_n = 1$ for all n and $\lim\limits_{t \to \infty} q_n = 1$. However, if $q_2 = 2$, then it is easy to show that

$$q = 2^{2^{n-2}}$$

for $n \geq 2$, so that $\{q_n\}$ diverges. This shows convergence is dependent on the starting values. In this example,

$$f(x, y) = x\gamma(y/x) = x^3/y^2.$$

x_1 and x_2 could be taken as $x_1 = x_2 = 1$ and $x_1 = 1$, $x_2 = 2$, respectively.

10. Example: Let $\gamma(t) = 1/t$ and let $q_2 = r > 0$ be arbitrarily chosen. Then

$$q_3 = \gamma\left(\frac{1}{q_2}\right) = r.$$

Similarly, $q_n = r$ for all $n \geq 4$. Therefore, $\{q_n\}$ is the constant sequence

$$\{r, r, \ldots, r, \ldots\},$$

which converges to r. In this example, it is seen that each positive real number is the limit of some q-sequence for the same generating function. Here,

$$f(x, y) = x\gamma\left(\frac{y}{x}\right) = \frac{x^2}{y}.$$

Starting values may be taken as $x_1 = 1$ and $x_2 = r > 0$.

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