ON ANDREWS' GENERALIZED FROBENIUS PARTITIONS

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1. Introduction

A generalized Frobenius partition or simply an F-partition of an integer n greater than 0 is a two-rowed array of nonnegative integers

$$\begin{pmatrix} a_1 & \dots & a_r \\ b_1 & \dots & b_r \end{pmatrix}$$

where each row is arranged in nonincreasing order and

$$n = r + \sum_{i=1}^{r} (a_i + b_i).$$

Let $c\phi_{k,h}(n)$ denote the number of those F-partitions of n in which each part is repeated at most h times and is taken from k copies of the nonnegative integers which are ordered as follows: $m_i < n_j$ if $m < n$ or if $m = n$ and $i < j$, where i and j denote the copy of the nonnegative integers. $c\phi_{k,h}(n)$ is called the number of F-partitions of n with k colors and h repetitions. Let $C\phi_{k,h}(q)$ be the generating function of $c\phi_{k,h}(n)$ so that

$$C\phi_{k,h}(q) = \sum_{n=0}^{\infty} c\phi_{k,h}(n)q^n.$$

For example, the F-partitions enumerated by $c\phi_{2,2}(1)$ are

$$\begin{pmatrix} 0_1 \\ 0_1 \end{pmatrix}\begin{pmatrix} 0_2 \\ 0_1 \end{pmatrix}\begin{pmatrix} 0_1 \\ 0_2 \end{pmatrix}\begin{pmatrix} 0_2 \\ 0_2 \end{pmatrix}$$

and those enumerated by $c\phi_{2,2}(2)$ are

$$\begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}\begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}\begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}\begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}\begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}$$

$$\begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}\begin{pmatrix} 0_2 & 0_2 \\ 0_2 & 0_2 \end{pmatrix}\begin{pmatrix} 0_2 & 0_2 \\ 0_2 & 0_1 \end{pmatrix}\begin{pmatrix} 0_2 & 0_2 \\ 0_1 & 0_1 \end{pmatrix}\begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_2 \end{pmatrix}$$

$$\begin{pmatrix} 0_2 & 0_1 \\ 0_1 & 0_1 \end{pmatrix}\begin{pmatrix} 0_1 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}\begin{pmatrix} 0_1 & 0_1 \\ 0_1 & 0_1 \end{pmatrix}\begin{pmatrix} 0_1 & 0_1 \\ 0_2 & 0_2 \end{pmatrix}$$

and

$$c\phi_{2,2}(q) = 1 + 4q + 17q^2 + \cdots .$$

Similarly,

$$C\phi_{3,2}(q) = 1 + 9q + 54q^2 + \cdots .$$

George E. Andrews [2] has studied extensively the two functions

$$c\phi_{1,k}(n) = \phi_k(n) \quad \text{and} \quad c\phi_{k,1}(n) = c\phi_k(n).$$

The former function enumerates the F-partitions of n in which the parts repeat at most k times and the latter enumerates those F-partitions of n in which the parts are distinct and are colored with k given colors. Andrews [2] has obtained infinite product representations for

$$C\phi_{1,1}(q) = \phi_1(q), \quad C\phi_{1,2}(q) = \phi_2(q),$$
$$C\phi_{1,3}(q) = \phi_3(q), \quad C\phi_{2,1}(q) = C\phi_2(q)$$

and has expressed $C\phi_{3,1}(q) = C\phi_3(q)$ as a sum of two infinite products. The purpose of this paper is to outline a method of obtaining such representations for $C\phi_{k,h}(q)$ for arbitrary positive integers k and h. We first consider in §2 the typical cases $C\phi_{2,2}(q)$ and $C\phi_{2,3}(q)$ and sketch in §3 how the methods of §2 can be extended for $C\phi_{k,h}(q)$ for arbitrary positive integers k and h. Throughout, we use the notations

$$(a)_\infty = (a, q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$$

for complex numbers a and q with $|q| < 1$.

2. Representations of $C\phi_{2,2}(q)$ and $C\phi_{2,3}(q)$

Theorem 1: For $|q| < 1$,

$$C\phi_{2,2}(q) = A_0(q)^2(q^4; q^4)_\infty(-q^2; q^4)_\infty^2 + 2q^{-1}[qB_0(q)]^2(q^4; q^4)_\infty(-q^4; q^4)_\infty^2, \tag{1}$$

where $A_0(q) = \phi_2(q)$ and $qB_0(q)$ is the generating function for symbols

$$\begin{pmatrix} \alpha_1 & \cdots & \alpha_r\alpha_{r+1} \\ \beta_1 & \cdots & \beta_r \end{pmatrix} \tag{2}$$

That is, this is subject to the same rules as the original generalized Frobenius symbols related to $\phi_2(q)$ but there is one more element in the top now. This sort of generalization of the Frobenius symbol has been studied at length by James Propp in a forthcoming article in the *Journal of Combinatorial Theory*.

Proof: To prove (1) we first make use of the following result of Andrews [3, Lemma 3]:

$$\begin{aligned}(z\alpha q)_\infty(z\beta q)_\infty(z^{-1}\alpha^{-1})_\infty(z^{-1}\beta^{-1})_\infty \\ = A_0(\alpha, \beta, q)\sum_{n=-\infty}^{\infty} q^{n^2+n}\alpha^n\beta^n z^{2n} \\ - \beta^{-1}A_0(\alpha q, \beta, q)\sum_{n=-\infty}^{\infty} q^{n^2}\alpha^n\beta^n z^{2n-1},\end{aligned} \tag{3}$$

where z, α, β are nonzero, $|q| < 1$, and

$$A_0(\alpha, \beta, q) = (-q)_\infty(-\alpha\beta^{-1}q; q^2)_\infty(-\alpha^{-1}\beta q; q^2)_\infty(q)_\infty^{-1}. \tag{4}$$

Choosing $\alpha = \omega$, $\beta = \omega^2$ in (3) where $\omega = \exp(2\pi i/3)$ and observing that

$$\prod_{n=1}^{\infty}(1 - q^{2n-1} + q^{4n-2}) = \frac{(-q^3; q^6)_\infty}{(-q; q^2)_\infty}$$

we obtain

$$\prod_{n=0}^{\infty}(1 + zq^{n+1} + z^2q^{2n+2})(1 + z^{-1}q^n + z^{-2}q^{2n}) \tag{5}$$
$$= A_0(q)\sum_{n=-\infty}^{\infty} q^{n^2+n}z^{2n} - B_0(q)\sum_{n=-\infty}^{\infty} q^{n^2}z^{2n-1},$$

where

$$A_0(q) = \frac{(-q^2;\ q^2)_\infty(-q^3;\ q^6)_\infty}{(q)_\infty} = \phi_2(q) \tag{6}$$

$$B_0(q) = -\frac{(-q;\ q^2)_\infty(-q^6;\ q^6)_\infty}{(q)_\infty}. \tag{7}$$

From the General Principle of Andrews [2, p. 5], it immediately follows that $C\phi_{2,2}(q)$ is the constant term in

$$\prod_{n=0}^{\infty}(1 + zq^{n+1} + z^2q^{2n+2})^2(1 + z^{-1}q^n + z^{-2}q^{2n})^2.$$

Squaring (5) and equating the constant terms, we get

$$C\phi_{2,2}(q) = \phi_2(q)^2\sum_{n=-\infty}^{\infty} q^{2n^2} + [qB_0(q)]^2\sum_{n=-\infty}^{\infty} q^{2n^2-2n-1}. \tag{8}$$

Now, using Jacobi's triple product identity [1, p. 21]:

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^n = (q^2;\ q^2)_\infty(-qz;\ q^2)_\infty(-qz^{-1};\ q^2)_\infty \tag{9}$$

for $z \neq 0$, $|q| < 1$ for the two summations in (8) we get (1).

From the proof of Theorem 1, it immediately follows that $C\phi_{2,2}(q)$ has the following representation.

Corollary 1: For $|q| < 1$,

$$C\phi_{2,2}(q) = \frac{(-q^2;\ q^2)_\infty^2(-q^3;\ q^6)_\infty^2(q^4;\ q^4)_\infty(-q^2;\ q^4)_\infty^2}{(q)_\infty^2} \tag{10}$$
$$+ 2q\,\frac{(-q;\ q^2)_\infty^2(-q^6;\ q^6)_\infty(q^4;\ q^4)_\infty(-q^4;\ q^4)_\infty^2}{(q)_\infty^2}.$$

Theorem 2: For $|q| < 1$,

$$C\phi_{2,3}(q) = A_1(q)^2(q^6;\ q^6)_\infty(-q^3;\ q^6)_\infty^2 \tag{11}$$
$$+ q^{-5}[qB_1(q)][q^2C_1(q)](q^6;\ q^6)_\infty(-q;\ q^6)_\infty(-q^5;\ q^5)_\infty,$$

where $A_1(q) = \phi_3(q)$, $qB_1(q)$ is the generating function for symbols (2) where a part can be repeated at most three times on each row and $q^2C_1(q)$ is the generating function for symbols

$$\begin{pmatrix}\alpha_1 & \cdots & \alpha_r\alpha_{r+1}\alpha_{r+2}\\ \beta_1 & \cdots & \beta_r\end{pmatrix}$$

which has two more elements in the top row than the original generalized Frobenius symbol.

Proof: Proof of (11) is similar to the proof of (1) and we give only a sketch. First, for α, β, γ, z nonzero and $|q| < 1$, we can obtain the Laurent expansion of the product

$$(z\alpha q)_\infty (z\beta q)_\infty (z\gamma q)_\infty (z^{-1}\alpha^{-1})_\infty (z^{-1}\beta^{-1})_\infty (z^{-1}\gamma^{-1})_\infty \tag{3'}$$

in the same way the analogous Andrews' identity (3) above is derived [3, Lemma 3]. Then, substituting $\alpha = i$, $\beta = -i$, and $\gamma = -1$ in that Laurent expansion, we obtain in analogy with (5)

$$\begin{aligned}\prod_{n=0}^{\infty} &(1 + zq^{n+1} + z^2q^{2n+2} + z^3q^{3n+3})(1 + z^{-1}q^n + z^{-2}q^{2n} + z^{-3}q^{3n}) \qquad (5')\\ &= \frac{(q^2;\, q^2)_\infty (q;\, q^2)_\infty^2 (q^6;\, q^6)_\infty}{(q)_\infty^3}\Big[(-q^3;\, q^6)_\infty^2 \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(3n^2+3n)} z^{3n}\\ &\quad + q(-q;\, q^6)_\infty (-q^5;\, q^6)_\infty \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(3n^2+5n)} z^{3n+1}\\ &\quad + q^3(-q^{-1};\, q^6)_\infty (-q^7;\, q^6)_\infty \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(3n^2+7n)} z^{3n+2}\Big]\\ &= A_1(q)\Sigma_1 + B_1(q)\Sigma_2 + C_1(q)\Sigma_3, \text{ say.}\end{aligned}$$

From the General Principle [2, p. 5], it is clear that $C\phi_{2,3}(q)$ is a constant term in

$$\prod_{n=0}^{\infty} (1 + zq^{n+1} + z^2q^{2n+2} + z^3q^{3n+3})^2 (1 + z^{-1}q^n + z^{-2}q^{2n} + z^{-3}q^{3n})^2.$$

Squaring (5′) and equating the constant terms, we find

$$C\phi_{2,3}(q) = A_1(q)^2 \sum_{n=-\infty}^{\infty} q^{3n^2} + [qB_1(q)][q^2C_1(q)] \sum_{n=-\infty}^{\infty} q^{3n^2+2n-5}. \tag{8'}$$

Finally, using (9) for the two summations in (8′), we obtain (11).

From the proof of Theorem 2, we obtain the following representation of $C\phi_{2,3}(q)$.

Corollary 2: For $|q| < 1$,

$$\begin{aligned}C\phi_{2,3}(q) &= \frac{(q^2;\, q^2)_\infty^2 (q;\, q^2)_\infty^4 (q^6;\, q^6)_\infty^3}{(q)_\infty^6}[(-q^3;\, q^6)_\infty^6 + \\ &\quad + 2q^2(-q;\, q^6)_\infty^2 (-q^5;\, q^6)_\infty^2 (-q^{-1};\, q^6)_\infty (-q^7;\, q^6)_\infty].\end{aligned} \tag{12}$$

3. Representation of $C\phi_{k,h}(q)$ in General

The representation of $C\phi_{k,h}(q)$ for arbitrary positive integers k and h is obtained in Theorem 3 in the same way we obtain the special cases (1) and (11), but after suitable generalizations of the methods. Lemma 1 furnishes a result which plays the role played by Jacobi's triple product identity in passing from (8) to (1) and from (8′) to (11). Due to the mechanical nature of the steps, we only sketch our proofs and avoid lengthy expressions.

Lemma 1: For $a > 0$, $a_1, \ldots, a_{k-1}$ integers and $|q| < 1$, the series

$$\sum_{n_1, \ldots, n_{k-1}=-\infty}^{\infty} q^{a\left(\sum_{i=1}^{k-1} n_i^2 + \sum_{1\le i<j\le k-1} n_i n_j\right) + \sum_{i=1}^{k-3} a_i n_i} \tag{9'}$$

can be expressed as a sum of $2^{k-2}3^{k-3}4^{k-4} \ldots (k-1)^{k-(k-1)}$ infinite products.

Proof: First Step. By grouping terms with $n_1, \ldots, n_{k-2}$ even and $n_1, \ldots, n_{k-2}$ odd separately, the series (9′) can be written as the sum of 2^{k-2} series, each of which will be of the form

$$q^{m_1} \sum_{n_1, \ldots, n_{k-2}=-\infty}^{\infty} q^{a\left(3n_1^2 + \cdots + 3n_{k-2}^2 + 2\sum_{1\le i<j\le k-2} n_i n_j\right) + \sum_{i=1}^{k-2} b_i n_i}$$
$$\times \sum_{n_{k-1}=-\infty}^{\infty} q^{an_{k-1}^2 + bn_{k-1} + b'},$$

where $m_1, b_1, \ldots, b_{k-2}, b, b'$ are integers. Here, the second series can be written as an infinite product by Jacobi's triple product identity (9). Thus, it suffices to express the first series as a product.

Second Step: By grouping terms with $n_1, \ldots, n_{k-3} \equiv r \pmod 3$, $r = 0, 1, 2$, the first series of the first step can be written as the sum of 3^{k-3} series, each of which will be of the form

$$q^{m_2} \sum_{n_1, \ldots, n_{k-3}=-\infty}^{\infty} q^{a\left(24n_1^2 + \cdots + 24n_{k-3}^2 + 12\sum_{1\le i<j\le k-3} n_i n_j\right) + \sum_{i=1}^{k-3} c_i n_i}$$
$$\times \sum_{n_{k-2}=-\infty}^{\infty} q^{3an_{k-2}^2 + cn_{k-2} + c'},$$

where $m_2, c_1, \ldots, c_{-3}, c, c'$ are all integers.

Proceeding similarly, we arrive at the $(k-2)^{\text{th}}$ step, namely,

$(k-2)^{th}$ Step: By grouping terms with $n_1 \equiv r \pmod{k-1}$, $r = 0, 1, 2, \ldots, k-2$, separately, the first series of the $(k-3)^{\text{th}}$ step can be written as a sum of $(k-1)^{k-(k-1)} = k-1$ series, each of which will be of the form

$$q^{m_{k-2}} \sum_{n_1=-\infty}^{\infty} q^{\alpha_1 n_1^2 + \beta_1 n_1 + \gamma_1} \sum_{n_2=-\infty}^{\infty} q^{\alpha_2 n_2^2 + \beta_2 n_2 + \gamma_2},$$

(where $m_{k-2}, \alpha_i, \beta_i, \gamma_i$ for $i = 1, 2$ are integers), which are explicit infinite products by (9).

Conclusion: From Steps 1 to $(k-2)$, it is clear that the series (9′) can be written as a sum of $2^{k-2}3^{k-3}4^{k-4} \ldots (k-1)$ infinite products. This proves Lemma 1.

Theorem 3: For arbitrary positive integers k and h, $C\phi_{k,h}(q)$ can be expressed as a sum of infinite products.

Proof: For $z, \alpha_1, \ldots, \alpha_h$ all nonzero and $|q| < 1$, we consider the product

$$(z\alpha_1 q)_\infty (z\alpha_2 q)_\infty \ldots (z\alpha_h q)_\infty (z^{-1}\alpha_1^{-1})_\infty (z^{-1}\alpha_2^{-1})_\infty \ldots (z^{-1}\alpha_h^{-1})_\infty, \tag{3''}$$

which, on using (9), can be written as

$$(q)_\infty^{-h} \sum_{n_1=-\infty}^{\infty} (-1)^{n_1} q^{\frac{n_1^2+n_1}{2}} \alpha_1^{n_1} z^{n_1} \ \ldots \ \sum_{n_h=-\infty}^{\infty} (-1)^{n_h} q^{\frac{n_h^2+n_h}{2}} \alpha_h^{n_h} z^{n_h}. \tag{13}$$

It is not difficult to realize a procedure for obtaining the Laurent expansion of the product (13). For instance, consider, for arbitrary integers a, b, c, d, e, f and nonzero z, α, β and $|q| < 1$, the product

$$\sum_{m=-\infty}^{\infty} q^{am^2+bm} \alpha^m z^{cm} \sum_{n=-\infty}^{\infty} q^{dn^2+en} \beta^n z^{fn}. \tag{14}$$

In this, let

$$x = \frac{c}{(c,\ f)} \quad \text{and} \quad y = \frac{f}{(c,\ f)}.$$

By grouping terms with $m \equiv r \pmod{y}$, $r = 0, 1, \ldots, y-1$, separately, and then changing n to $n - xm$, (14) can be written as sum of y number of series of the form

$$\sum_{m=-\infty}^{\infty} S(\beta,\ z,\ q,\ m) \sum_{n=-\infty}^{\infty} q^{an^2-bmn+cn} \alpha^n. \tag{15}$$

Now grouping terms with $m \equiv r \pmod{e}$, $r = 0, 1, \ldots, e-1$, where $2ad - be = 0$ with $(d, e) = 1$ in (15), we obtain the Laurent expansion of (14).

By applying the above procedure successively, we obtain the Laurent expansion of (13). Substituting $\alpha_1 = \omega, \ldots, \alpha_h = \omega^h$, where $\omega = \exp(2\pi i/h + 1)$ in that Laurent expansion, multiplying the resulting identity k times, and equating the constant terms, we find $C\phi_{k,h}(q)$ to be a sum of series of the form (9′) which, by Lemma 1, is a sum of $2^{k-2}3^{k-3} \ldots (k-1)$ infinite products.

It would be interesting to obtain combinatorial proofs of equations (8) and (8′) which might throw more light on this subject.

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