FIBONACCI NUMBERS AND BIPYRAMIDS

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(Submitted June 1987)

1. Introduction

A *bipyramid* B_n of order $n \geq 5$ with degree sequence

$$d_1 \leq d_2 \leq \cdots \leq d_n, \; d_{n-1} = d_n = n - 2$$

is a maximal planar graph consisting of a cycle of order $n - 2$ and two nonadjacent vertices u and v. Every vertex of the cycle has degree 4 and is adjacent to both u and v whose degrees are $n - 2$ as in Figure 1.

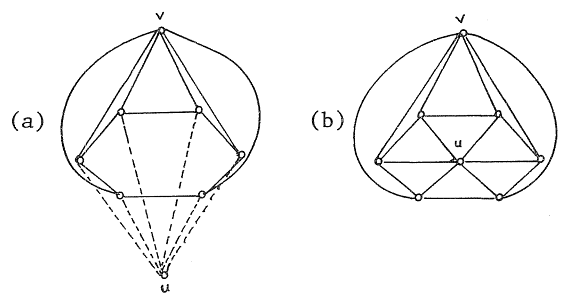

FIGURE 1. A bipyramid with $n = 8$

If B_n is redrawn as in Figure 1(b), then it is geometrically obvious that all such maximal planar graphs contain wheels as subgraphs with $n - 2$ vertices on the rim and a center u with degree $n - 2$ [3]. The graph B_n is called a *generalized bipyramid* if the restriction on d_{n-1} is relaxed while preserving maximal planarity with $3 \leq d_{n-1} \leq n - 2$. Some maximal planar graphs B_8 are shown in Figure 2.

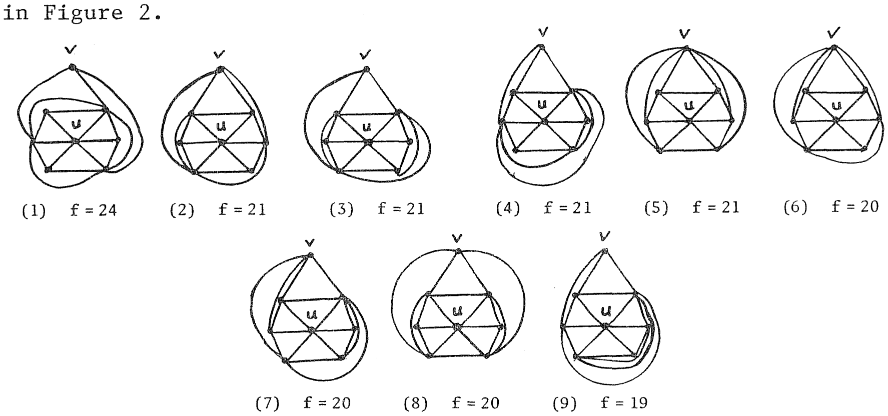

FIGURE 2. Some bipyramids B_n of order 8

The Fibonacci number $f(G)$ of a simple graph G is the number of all complete subgraphs of the complement graph of G. In this paper, our main goal is to present a structural characterization of the class of generalized bipyramids whose Fibonacci numbers are minimum. We will prove that, if G is a maximal planar graph of order n belonging to this class, then

$$f(G) \sim (0.805838\ldots)(1.465571\ldots)^n.$$

This result will be achieved via outerplanar graphs.

Prodinger and Tichy [2] gave upper and lower bounds for trees: If T is a tree on n vertices, then

$$F_{n+1} \le f(T) \le 2^{n-1} + 1,$$

where F_n is the n^{th} Fibonacci number of the sequence

$$F_n = F_{n-1} + F_{n-2},\ F_0 = F_1 = 1.$$

The upper and lower bounds are assumed by the stars S_n and paths P_n in Figure 3, where

$$f(S_n) = 2^{n-1} + 1 \quad \text{and} \quad f(P_n) = F_{n+1}.$$

The upper bound of the set of all maximal outerplanar graphs was investigated in [1]. It is shown that if G is a maximal outerplanar graph of order n and N_n is the fan shown in Figure 3, then

$$f(G) \le f(N_n) = F_n + 1.$$

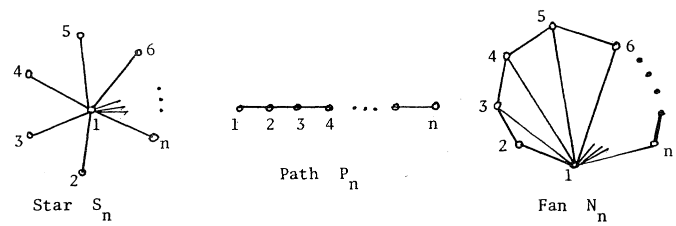

Figure 3. Stars, Paths, and Fans

2. From Maximal Planar to Maximal Outerplanar

From the definition of the Fibonacci number of a graph, we observe that the number of complete subgraphs in the complement of B_n is the same as the number of those complete subgraphs that do not contain the center u and the number of those that do contain u. That is,

$$f(B_n) = f(B_n - u) + 2.$$

The graphs $B_n - u$ for $n = 8$ are redrawn in Figure 4.

Let $C_{n-1} = B_n - u$ and consider the vertex v in C_n. We have

$$f(C_{n-1}) = f(H_{n-2}) + f(H'_{n-2}),$$

where $f(H_{n-2})$ is the number of complete subgraphs in the complement of $C_{n-1} - v$ and $f(H'_{n-2})$ is the number of those complete subgraphs of the complement of C_{n-1} that contain v. We remark that if an edge e is added to two nonadjacent vertices of any graph G without destroying maximal planarity, then

$$f(G) > f(G + e);$$

e is called a *chord* if it is not a rim edge. It suffices to show that the graph

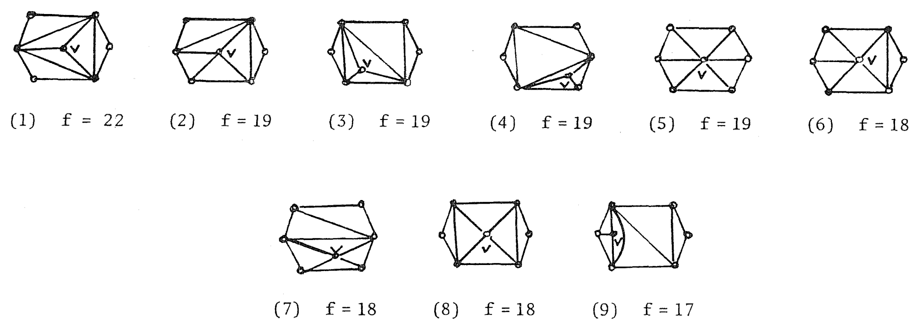

FIGURE 4. Fibonacci numbers of $B_n - u$, $n = 8$

C_{n-1} has minimum f if the remaining chords form longest paths in H_{n-2} and H'_{n-2} as in graph (9) in Figures 4 and 5. That is, $f(C_{n-1})$ is minimum if both H_{n-2} and H'_{n-2} are maximal outerplanar graphs with longest paths of chords.

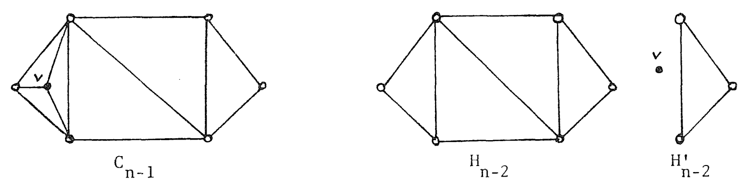

FIGURE 5. $f(C_{n-1}) = f(H_{n-2}) + f(H'_{n-2})$

Since a maximal outerplanar graph G is a triangulation of a polygon and every such graph has two vertices of degree two, there are two triangles T_1 and T_2 in G each of which has a vertex of degree 2. If the vertex v is chosen in one of these triangles, then we have the following theorem.

Theorem 1: Let H_n be a maximal outerplanar graph of order n with a longest path of chords. Let $C_{n+1} = H_n + v$, where v is inserted in any triangle of H_n and joined to the corresponding vertices, then $f(C_{n+1})$ is minimum if $v \in T_1$ or $v \in T_2$.

Proof: Consider the formula

$$f(C_{n+1}) = f(H_n) + f(H'_n).$$

$f(H_n)$ is invariant under all possible choices of triangles, whereas H'_n has the same Fibonacci number if and only if $v \in T_1$ or $v \in T_2$. For all other choices of triangles, H'_n is a disjoint subgraph and hence has a larger Fibonacci number. □

In the next theorem, we show that among all maximal outerplanar graphs of the same order $f(H_n)$ is smallest.

Theorem 2: Let G be an arbitrary maximal outerplanar graph of order n. Then $f(H_n) \le f(G)$, where H_n is maximal outerplanar with longest path of chords.

Proof: Let G and H_n have the same order n and proceed by induction on n. Assume that $f(H_k) \le f(G)$ for all maximal outerplanar graphs G of order $k < n$.

Using the same labeling of the hamiltonian circuit of G we draw the graph H_n. This means that G and H_n differ only in the arrangements of the chords. Let u and v be vertices of degree 2 in G and H_n, respectively. Define $G^* = G - u$ and $H^* = H_n - v$. That is, G^* and H^* are the maximal outerplanar graphs of order $n - 1$ obtained by deleting u and v from G and H_n, respectively. Also, let G^{**} and H^{**} be the graphs obtained by deleting the two neighbors of u from G and the two neighbors of v from H_n. [Let $v = 2k$ in Figure 6(a) and $v = k$ in Figure 6(b).] We observe that the number of complete subgraphs in the complement of G is the sum of the number of those complete subgraphs which do not contain the vertex u and the number of those which do contain u. After noting that

$$f(G^{**}) = f(G^{**} - u),$$

we have

$$f(G) = f(G^*) + f(G^{**}) \quad \text{and} \quad f(H_n) = f(H^*) + f(H^{**}). \tag{1}$$

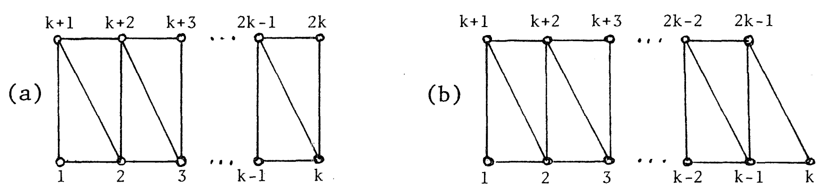

FIGURE 6. The graphs H_{2k} and H_{2k-1} with longest path of chords

Since G^* and H^* are maximal outerplanar of order $n - 1$, then, by the induction assumption,

$$f(H^*) \le f(G^*). \tag{2}$$

As for H^{**} and G^{**}, we see that the former is maximal outerplanar after deleting v (see Figure 6) while the latter need not be. However, by arbitrarily adding edges to $G^{**} - u$, we see that at each stage the Fibonacci number is less than that at the previous stage until we construct a maximal outerplanar graph G^{***} with $2(n - 3) - 3$ edges having $G^{**} - u$ as a subgraph (see Figure 7).

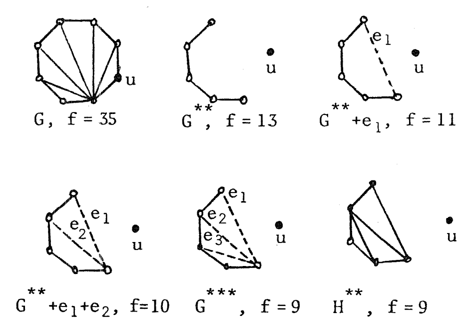

FIGURE 7. The construction of G^{***}, $n = 8$

Now, since $f(G^{**}) = f(G^{**} - u)$, we have

$$f(G^{***}) = f(G^{***} - u),$$

and since $H^{**} - u$ and G^{***} satisfy the hypotheses of the theorem and their order is less than n, we have

$$f(H^{**}) \le f(G^{***}) \le f(G^{**}). \tag{3}$$

From (1), (2), and (3), we see that $f(H_n) \le f(G)$ and the proof is complete. □

Now we show that these graphs H_n are the only ones with the relevant property.

Theorem 3: If G is a maximal outerplanar graph of order n with $f(G) = f(H_n)$, then G is isomorphic to H_n.

Proof: We argue by induction, assuming the result for small values. The argument for Theorem 2 shows that $f(G^*) = f(H_{n-1})$ and $f(G^{**}) = f(H_{n-3})$, where $f(G) = f(G^*) + f(G^{**})$. Hence, by the induction hypothesis, $G^* \simeq H_{n-1}$ and G^{**} is maximal outerplanar (by observing that an additional edge decreases the Fibonacci number) and is isomorphic to H_{n-3}. These conditions easily force the conclusion. □

3. The Fibonacci Number of H_n

The graphs H_n shown in Figure 6 satisfy the recurrence relation

$$h_n = h_{n-1} + h_{n-3}, \tag{4}$$

where $f(H_n) = h_n$, $h_0 = 1$, $h_1 = 2$, $h_2 = 3$.

The solution of (4) is

$$h_n = \left[\frac{u + v + 10}{3u + 3v}\right]\left[\frac{u + v + 1}{3}\right]^n + \left[\frac{u + v - 5}{3u + 3v}\right]\left[-\frac{u + v - 2}{6} + \frac{u - v}{6}\sqrt{3}\,i\right]^n + \left[\frac{u + v - 5}{3u + 3v}\right]\left[-\frac{u + v - 2}{6} - \frac{u - v}{6}\sqrt{3}\,i\right]^n,$$

where $u = \sqrt[3]{\dfrac{29 + 3\sqrt{93}}{2}}$ and $v = \sqrt[3]{\dfrac{29 - 3\sqrt{93}}{2}}$.

Since $f(C_{n-1}) = f(H_{n-2}) + f(H'_{n-2})$, we have

$$f(C_{n-1}) = f(H_{n-2}) + f(H_{n-5}) \quad \text{and} \quad f(B_n) = f(H_{n-2}) + f(H_{n-5}) + 2,$$

from which we can prove the following result.

Theorem 4: If B_n is the generalized bipyramid with minimum Fibonacci number, then

$$f(B_n) \sim c\alpha^n, \text{ where } c \approx 0.805838\ldots \text{ and } \alpha \approx 1.465571\ldots .$$

Proof: The order of growth of $f(H_n)$ is governed by the dominant root

$$\alpha = \frac{u + v + 1}{3}$$

and $f(H_n) \sim c_1\alpha^n$, where $c_1 \approx 1.3134\ldots$.

For the bipyramids B_n with minimum Fibonacci number, we have

$$f(B_n) = f(H_{n-2}) + f(H_{n-5}) + 2,$$

which implies

$$f(B_n) \sim c_1[\alpha^{n-2} + \alpha^{n-5}] \quad \text{or} \quad f(B_n) \sim c_1(\alpha^{-2} + \alpha^{-5})\alpha^n.$$

So, we can write

$$f(B_n) \sim (0.805838\ldots)\alpha^n, \text{ where } \alpha = 1.465571\ldots .$$

We summarize our results for small graphs and compare with F_n, $n \le 20$, in Table 1.

TABLE 1

Fibonacci numbers of various graphs of order ≤ 20

n	F_n	$f(N_n)$	$f(H_n)$	$f(B_n)$
0	1	1	1	
1	1	2	2	
2	2	3	3	
3	3	4	4	
4	5	6	6	
5	8	9	9	7
6	13	14	13	10
7	21	22	19	14
8	34	35	28	19
9	55	56	41	27
10	89	90	60	39
11	144	145	88	56
12	233	234	129	81
13	377	378	189	118
14	610	611	277	172
15	987	988	406	251
16	1597	1598	595	367
17	2584	2585	872	537
18	4181	4182	1278	786
19	6765	6766	1873	1151
20	10946	10947	2745	1686

Acknowledgment

The author wishes to acknowledge the support of King Fahd University of Petroleum & Minerals.

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