CHARACTERIZATIONS AND EXTENDIBILITY OF P_t-SETS

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Let t be a nonzero integer and S be a set of three or more integers. We will say that S is a P_t-set if, for any two distinct elements x and y of S, the integer $xy + t$ is a perfect square. A P_t-set S will be termed extendible if, for some integer d, $d \notin S$, the set $S \cup \{d\}$ is a P_t-set.

The purpose of this paper is to characterize certain families of P_t-sets, and to show that some of these are not extendible. In particular, the result of Thamotherampillai [1], that the P_2-set $\{1, 2, 7\}$ is not extendible, will be obtained as an easy corollary.

To simplify the exposition, throughout this paper statements of congruences are to be interpreted modulo 4; i.e., $x \equiv y$ will mean $x \equiv y \pmod 4$.

Lemma: If S is a P_t-set and $a, b, c \in S$, then none of the numbers

$$a(c - b), \quad b(c - a), \quad c(b - a)$$

is congruent to 2, modulo 4.

Proof: By the definition of P_t-sets, we have

$$ab + t = x^2, \quad ac + t = y^2, \quad bc + t = z^2$$

for some integers x, y, and z. Upon eliminating t among the equations above, the result follows from the fact that perfect squares are congruent to 0 or 1, modulo 4.

Theorem 1: If all of the elements of a P_t-set are odd, then they are congruent to one another, modulo 4.

Proof: Let S be a P_t-set, and $a, b, c \in S$. Observe that, if $a \equiv b \equiv 1$ and $c \equiv 3$, then $a(c - b) \equiv 2$; while if $a \equiv 1$ and $b \equiv c \equiv 3$, then $b(c - a) \equiv 2$. Both of these conclusions are impossible in view of the Lemma; hence, either $a \equiv b \equiv c \equiv 1$ or $a \equiv b \equiv c \equiv 3$.

Theorem 2: If only one of the elements of a P_t-set is odd, then all of the others are congruent to 0, modulo 4.

Proof: Let S be a P_t-set, and $a, b, c \in S$. Observe that, if $a \equiv 1$, $b \equiv 2$, and $c \equiv 0$ or if $a \equiv 3$, $b \equiv 2$, $c \equiv 0$, then $a(c - b) \equiv 2$; while if $a \equiv 1$ and $b \equiv c \equiv 2$ or if $a \equiv 3$ and $b \equiv c \equiv 2$, then $c(b - a) \equiv 2$. Both of these conclusions are impossible in view of the Lemma; hence, if $a \equiv 1$ or 3, then $b \equiv c \equiv 0$.

Theorem 3: P_t-sets of the form $\{4k + 1, 4m + 2, 4n + 3\}$ are not extendible.

Proof: Assume that $\{4k + 1, 4m + 2, 4n + 3, d\}$ is a P_t-set. If d is odd, then $\{4k + 1, 4n + 3, d\}$ is a P_t-set all of whose elements are odd. However, $4k + 1 \not\equiv 4n + 3$, contrary to Theorem 1. If d is even, then $\{4k + 1, 4m + 2, d\}$ is a P_t-set with only one odd element, $4k + 1$. But $4m + 2 \not\equiv 0$, contrary to Theorem 2. Consequently, such d cannot exist.

Corollary: The P_2-set $\{1, 2, 7\}$ is not extendible.

At this point, the authors wish to express their appreciation to Bud Brown, who sent them a copy of [2] upon reading [3], and hence called their attention to [1]. It may also be noted that Thamotherampillai's proof of the corollary is much more complicated, and its method does not allow for generalizations.

In conclusion, we provide a table of examples which shows that all of the cases not disallowed by Theorems 1 and 2 are indeed possible. In the "congruence type" column, the members of S are reduced modulo 4 to allow for a quick review; thus, for example, the P_{97}-set $\{3, 8, 24\}$ is type [3,0,0] since $3 \equiv 3$ and $8 \equiv 24 \equiv 0$. In this terminology, P_t-sets of types [1,1,3] and [1,3,3] do not exist in view of Theorem 1, P_t-sets of types [1,2,2], [1,2,0], [3,2,2], and [3,2,0] do not exist in view of Theorem 2; and P_t-sets of type [1,2,3] are not extendible in view of Theorem 3.

Table of Examples

Congruence type	S	t
[1,1,1]	{1,5,33}	31
[3,3,3]	{7,11,23}	323
[1,0,0]	{5,8,16}	41
[3,0,0]	{3,8,24}	97
[0,0,0]	{4,12,32}	16
[2,0,0]	{2,12,420}	1
[2,2,0]	{2,6,16}	4

Congruence type	S	t
[2,2,2]	{2,10,22}	5
[1,1,0]	{1,9,20}	16
[1,1,2]	{1,5,10}	-1
[1,3,0]	{1,7,16}	9
[1,3,2]	{1,79,98}	2
[3,3,0]	{3,27,60}	144
[3,3,2]	{3,7,2}	-5

References

1. N. Thamotherampillai. "The Set of Numbers {1, 2, 7}." *Bull. Calcutta Math. Society* 72 (1980):195-197.
2. Ezra Brown. "Sets in Which $xy + k$ is Always Square." *Math. Comp.* 45.172 (1985):613-620.
3. G. Berzsenyi. "Problems, Puzzles and Paradoxes: Discoveries." *Consortium* No. 25 (March 1988):5.

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